

Uniqueness of the first eigenfunction for fully nonlinear equations: the radial case.

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1 Introduction

The extension of the concept of eigenvalue to non-linear operator was started, in the variational case, to study existence of solutions for Dirichlet problems for operators such as the p -Laplacian and it has been proved a very fruitful field of research (see e.g. [1, 2, 20, 14]). In particular the simplicity of the first eigenvalue for the p -Laplacian was proved by Anane [1] and Ôtani and Teshima [21].

Very recently, inspired by the seminal result of Berestycki, Nirenberg and Varadhan [3], the concept of non-linear eigenvalue has been extended to elliptic, fully-nonlinear operators in non divergence form and it has been the object of many interesting papers. In particular we should mention the works of Busca, Esteban, Quaas [9], and Quaas [22] for the Pucci operators, the papers of Ishii, Yoshimura [17] and Quaas, Sirakov [23] for more general fullynonlinear uniformly elliptic operators which are homogeneous of degree 1 in the Hessian and degree zero on the gradient.

The authors of this note have defined the "principal eigenvalue" for fully-nonlinear degenerate or singular elliptic operators modelled on the p -Laplacian but not variational i.e. which are homogenous of degree $\alpha > -1$ in the gradient, see [4, 5, 6]. In those papers we prove the existence of the corresponding eigenfunction together with many other properties (regularity of the viscosity solutions, maximum principle, existence of solutions below the principal eigenvalue...). But in those papers we raised the question of whether the principal eigenfunction is unique up to multiplication by a constant. Here we answer this question when the eigenfunctions are radial.

We also wish to mention the work of Petri Juutinen [18] who treats even

more degenerate operators since he defines the principal eigenvalue for the infinite Laplacian using techniques somehow related with those used in [5].

We start by describing the general class of operators that we consider. Let $\alpha > -1$ and let S be the set of symmetric matrices $N \times N$. Let us suppose that b and c are continuous and bounded functions in Ω and let us consider

$$F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) |u|^\alpha u$$

with $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S$, satisfying homogeneity, ellipticity and some standard continuity assumptions, and with b satisfying some Hölder's continuity condition. All the hypothesis will be made precise in the next section.

In this class of operators one can consider for example the p -Laplacian or

$$F(x, p, X) = |p|^\alpha \mathcal{M}_{a,A}(X)$$

where $\mathcal{M}_{a,A}(X)$ is a Pucci operator.

In [6, 5] we showed that

$$\bar{\lambda}(\Omega) = \sup\{\lambda, \exists \varphi > 0 \text{ in } \Omega, F(x, \nabla \varphi, D^2 \varphi) + b(x) \cdot \nabla \varphi |\varphi|^\alpha + (c(x) + \lambda) |\varphi|^\alpha \varphi \leq 0 \text{ in } \Omega\}$$

and

$$\underline{\lambda}(\Omega) = \sup\{\lambda, \exists \varphi < 0 \text{ in } \Omega, F(x, \nabla \varphi, D^2 \varphi) + b(x) \cdot \nabla \varphi |\varphi|^\alpha + (c(x) + \lambda) |\varphi|^\alpha \varphi \geq 0 \text{ in } \Omega\}$$

are two eigenvalues in the following sense:

There exists an eigenfunction $\varphi > 0$ such that in the viscosity sense

$$\begin{cases} F(x, \nabla \varphi, D^2 \varphi) + b(x) \cdot \nabla \varphi |\nabla \varphi|^\alpha + (c(x) + \bar{\lambda}) |\varphi|^\alpha \varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega; \end{cases}$$

and there exists $\psi < 0$ such that in the viscosity sense:

$$\begin{cases} F(x, \nabla \psi, D^2 \psi) + b(x) \cdot \nabla \psi |\nabla \psi|^\alpha + (c(x) + \underline{\lambda}) |\psi|^\alpha \psi = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

One way of characterizing the eigenvalue $\bar{\lambda}$ is that for any $\lambda < \bar{\lambda}$ the maximum principle holds, and, analogously, for any $\lambda < \underline{\lambda}$ the minimum principle holds

(see [5],[6]). Clearly since the operators are highly nonlinear these properties do not imply the validity of some strict comparison principle which is tightly linked to the question of the simplicity of the eigenfunction and of the isolation of the eigenvalue which was raised in [5],[6].

Since here we prove the simplicity of radial eigenfunctions, we suppose that Ω is rotationally invariant i.e. up to a translation it is either a ball or an annular region centered at the origin.

The key ingredients for obtaining these results are the Hopf principle, a strict comparison principle near the boundary, and specific properties in the radial case.

Let us note that the classical approach cannot be taken because it is not known if the Alexandrov Bakelman Pucci inequality holds true for the solutions of the class of equations treated here.

2 Notations and hypothesis

We begin by detailing the hypothesis on the operator $F : \Omega \times (\mathbb{R}^N) \setminus \{0\} \times S \rightarrow \mathbb{R}$.

Let $\alpha > -1$ and let S be the set of symmetric matrices $N \times N$.

(H1) For all $x \in \Omega$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$, $\forall t \in \mathbb{R} \setminus \{0\}$, $\mu \in \mathbb{R}^+$.

(H2) $\exists a, A > 0$ such that for all $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$ and $(M, N) \in S^2$, $N \geq 0$:

$$a|p|^{\alpha \text{tr} N} \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^{\alpha \text{tr} N}$$

(H3) There exists a continuous function $\tilde{\omega}$, $\tilde{\omega}(0) = 0$ such that for all $(x, y) \in \Omega^2$, for all $p \neq 0$ and for all $X \in S$

$$|F(x, p, X) - F(y, p, X)| \leq \tilde{\omega}(|x - y|)|p|^\alpha |X|.$$

(H4) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in \mathbb{R}^N , then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$F(x, \zeta(x-y), X) - F(y, \zeta(x-y), -Y) \leq \omega(\zeta|x-y|^2).$$

Concerning b we assume that $b : \Omega \mapsto \mathbb{R}^N$ is a continuous and bounded function satisfying:

- (H5) -Either $\alpha < 0$ and b is Hölder of exponent $1 + \alpha$,
 -or $\alpha \geq 0$ and, for all x and y in Ω , $\langle b(x) - b(y), x - y \rangle \leq 0$.

In particular the condition (H1) implies that if $\phi(x) = g(|x|)$ then

$$F(x, \nabla \phi, D^2(\phi)) = |g'(|x|)|^\alpha F(x, \frac{x}{|x|}, g'' \frac{x}{|x|} \otimes \frac{x}{|x|} + \frac{g'}{|x|} (I - \frac{x}{|x|} \otimes \frac{x}{|x|})).$$

Considering now condition (H2), it implies that

$$|g'|^\alpha \left(\gamma_1 g'' + \frac{\gamma_2(n-1)}{|x|} g' \right) \leq F(x, \nabla \phi, D^2(\phi)) \leq |g'|^\alpha \left(\Gamma_1 g'' + \frac{\Gamma_2(n-1)}{|x|} g' \right)$$

where

$$\gamma_1 = \begin{cases} a & \text{if } g'' > 0 \\ A & \text{if } g'' < 0 \end{cases}, \quad \gamma_2 = \begin{cases} a & \text{if } g' > 0 \\ A & \text{if } g' < 0, \end{cases}$$

$$\Gamma_1 = \begin{cases} A & \text{if } g'' > 0 \\ a & \text{if } g'' < 0 \end{cases}, \quad \Gamma_2 = \begin{cases} A & \text{if } g' > 0 \\ a & \text{if } g' < 0, \end{cases}$$

see [15] for a similar computation.

2.1 Radial eigenfunctions

In the rest of the paper we suppose that F , b and c are such that there exists an eigenfunction ϕ corresponding to $\bar{\lambda}$ which is radial, i.e. $\phi(x) = g(|x|)$ for some real function g , and $\Omega = B(0, 1)$ or $\Omega = B(0, 1) \setminus B(0, \rho)$.

For completeness sake, let us mention that it is the case when $b(x) \cdot \frac{x}{|x|} = h(|x|)$ for some real function h , $c(x) = c(|x|)$ and there exists $\tilde{F} : \mathbb{R}^{+,*} \times \mathbb{R}^* \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for $r = |x|$,

$$|g'(r)|^\alpha F(x, g'(r) \frac{x}{r}, g'' \frac{x}{r} \otimes \frac{x}{r} + \frac{g'}{r} (I - \frac{x}{r} \otimes \frac{x}{r})) = \tilde{F}(r, g', g'').$$

In this situation we can define

$$\bar{\lambda}_r = \sup\{\lambda; \exists g > 0, \tilde{F}(r, g', g'') + h(r)g'|g'|^\alpha + (c(r) + \lambda)g^{1+\alpha} \leq 0, \text{ in } \Omega\}.$$

Following the arguments in [5], one can prove that there exists $g > 0$ in Ω solution of

$$\begin{cases} \tilde{F}(r, g', g'') + h(r)g'|g'|^\alpha + (c(r) + \lambda_r)g^{1+\alpha} = 0 & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

In particular this implies that $\bar{\lambda}_r = \bar{\lambda}(\Omega)$; indeed, by definition, $\bar{\lambda}_r \leq \bar{\lambda}(\Omega)$, but if $\bar{\lambda}_r < \bar{\lambda}(\Omega)$, then by the maximum principle this would imply that the above solution g would be strictly negative, a contradiction.

We now introduce some notations for left and right "derivatives" that will be useful in the rest of the paper:

Definition 2.1 *Let us recall the definition of the four number derivatives :*

$$\begin{aligned} d_l u(\bar{r}) &= \liminf_{h \rightarrow 0, h < 0} \frac{u(\bar{r} + h) - u(\bar{r})}{h} \\ d_r u(\bar{r}) &= \liminf_{h \rightarrow 0, h > 0} \frac{u(\bar{r} + h) - u(\bar{r})}{h} \\ D_l u(\bar{r}) &= \limsup_{h \rightarrow 0, h < 0} \frac{u(\bar{r} + h) - u(\bar{r})}{h} \\ D_r u(\bar{r}) &= \limsup_{h \rightarrow 0, h > 0} \frac{u(\bar{r} + h) - u(\bar{r})}{h} \end{aligned}$$

Remark 2.2 *According to the regularity results obtained in [5], the solutions of the Dirichlet problem*

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) |u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.2)$$

are Lipschitz inside Ω . This in particular implies that for solutions of (2.2) the limits given in the above definition are finite.

Proposition 2.3 *Let u be a radial Lipschitz supersolution of (2.2) with f bounded by above, then, for every $\bar{r} \in]\rho, 1[$,*

$$D_l u(\bar{r}) \geq d_r u(\bar{r}).$$

Similarly if u is a Lipschitz sub-solution of (2.2) with f bounded by below, then, for every $\bar{r} \in]\rho, 1[$,

$$D_r u(\bar{r}) \geq d_l u(\bar{r}).$$

Remark 2.4 *Since u is Lipschitz, almost everywhere the four number derivatives coincide.*

Proof. Suppose by contradiction that

$$D_l u(\bar{r}) < d_r u(\bar{r})$$

and let p be in $]D_l u(\bar{r}), d_r u(\bar{r})[$ then for every $q \in \mathbb{R}$, $\varphi(r) = u(\bar{r}) + p(r - \bar{r}) + q(r - \bar{r})^2$ touches u by below on \bar{r} and then, for some value C depending only on p and the data but not on q , one would have, for $q \geq 0$

$$(aq - \frac{A(N-1)p}{\bar{r}})|p|^\alpha + C \leq f.$$

A contradiction for q large since f is bounded by above. The analogous result for subsolution is easy to prove in the same manner.

We now give a definition

Definition 2.5 *For $r \neq 0$, we shall say, in what follows, that $u'(r) \not\approx 0$ if*

$$\text{either } \inf(D_l u(r), D_r u(r)) > 0 \quad \text{or} \quad \sup(d_l u(r), d_r u(r)) < 0.$$

While $u'(r) \not\approx 0$ means that

$$D_l u(r).D_r u(r) \leq 0 \quad \text{and} \quad d_l u(r).d_r u(r) \geq 0.$$

Proposition 2.6 *Suppose that u satisfies $u'(\bar{r}) \not\approx 0$. Then for every test function φ by above or below on \bar{r} , $|\varphi'(\bar{r})| \geq \inf(\inf(D_l u(\bar{r}), D_r u(\bar{r})), |\sup(d_l u(\bar{r}), d_r u(\bar{r}))|)$*

Proof. Suppose for example that $D_l u(\bar{r})$ and $D_r u(\bar{r})$ are both strictly positive. Let φ be such that φ touches u by below on \bar{r} . Then for $r < \bar{r}$

$$u(r) - u(\bar{r}) \geq \varphi(r) - \varphi(\bar{r})$$

and then dividing by $r - \bar{r} < 0$ one gets

$$\frac{\varphi(r) - \varphi(\bar{r})}{r - \bar{r}} \geq \frac{u(r) - u(\bar{r})}{r - \bar{r}}.$$

Taking the limsup on both sides and using the fact that φ is differentiable one gets that

$$\varphi'(\bar{r}) \geq D_l u(\bar{r}).$$

In the same manner if φ touches u by above for $r > \bar{r}$ one has

$$u(r) - u(\bar{r}) \leq \varphi(r) - \varphi(\bar{r}),$$

dividing by $r - \bar{r} > 0$ one gets

$$\frac{\varphi(r) - \varphi(\bar{r})}{r - \bar{r}} \geq \frac{u(r) - u(\bar{r})}{r - \bar{r}},$$

and taking the limsup on both sides one gets

$$\varphi'(\bar{r}) \geq D_r u(\bar{r}).$$

Analogous arguments permit to prove the other cases.

Proposition 2.7 *Suppose that u is a nonnegative, radial, continuous, nontrivial, supersolution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) u^{1+\alpha} \leq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $c > 0$ in Ω .

If $\Omega = B(0, 1)$ then 0 is a maximum point for u and it is the only point on which u' is zero.

If $\Omega = B(0, 1) \setminus \overline{B(0, \rho)}$ there exists at most one value r_u on which u achieves its maximum, which is also the only point on which u' is zero.

Proof . Let us start by observing that for $c > 0$, any non-negative super solution, which is not identically zero, cannot be locally constant.

Secondly, let us observe that any non-negative radial super-solution reaches its maximum in one point only. Indeed if $u(r_a) = u(r_b) = \max u(r)$ with $r_a < r_b$ then, since the positive constants are strict subsolutions, using the comparison principle (see Theorem 1 of [6]) one would have that $u(r) \geq \max u(r)$ for $r \in (r_a, r_b)$, contradicting the first observation.

In the case $\Omega = B(0, 1) \setminus \overline{B(0, \rho)}$, we can choose any $t \in]0, \max u[$. By the continuity of u there exist r_1 and r_2 such that $\rho < r_1 < r_u, 1 > r_2 > r_u$ such that $u(r_1) = u(r_2) = t$. By the comparison principle on the set $B(0, r_2) \setminus \overline{B(0, r_1)}$, reasoning as before one has

$$u(r) \geq t, \text{ for any } r \in B(0, r_2) \setminus \overline{B(0, r_1)}.$$

Moreover the minimum is achieved on the boundary of $B(0, r_2) \setminus \overline{B(0, r_1)}$ so, since u is nowhere locally constant, $d_r u(r_1) > 0$ and $D_l u(r_2) < 0$ by Hopf principle (see Corollary 1 of [6]).

In the case $\Omega = B(0, 1)$ reasoning as above we get that the maximum point has to be 0, and for any $t \in]0, \max u[$ by the continuity of the super solutions there exists $r_2 \in (0, 1)$ such that $u(r_2) = t = \inf_{B(0, r_2)} u$ and $D_l u(r_2) > 0$.

In both cases, by Proposition 2.3 we know that $D_l u(r_1) \geq d_r u(r_1) > 0$ and then $\inf(D_l u(r_1), D_r u(r_1)) > 0$ i.e., according to Definition 2.5, $u'(r_1) \not\approx 0$.

Similarly, concerning r_2 , one has $0 > D_l u(r_2) \geq d_r u(r_2)$ and then

$$\inf(d_l u(r_2), d_r u(r_2)) < 0,$$

which means that $u'(r_2) \not\approx 0$.

Since t was chosen arbitrarily, we can conclude that r_u is the only point on which u' can be zero.

Remark 2.8 *The previous result establishes that $u'(\bar{r}) \not\approx 0$ as soon as $\bar{r} \neq r_u$. Without the information that the left and right derivatives defined above satisfy some kind of continuity, one cannot conclude that there exists some constant $m > 0$ such that*

$$\inf(D_r u(r), D_l u(r)) \geq m$$

on every compact subset of $] \rho, r_u [$. This will be a necessary ingredient in what follows, and it is the object of the following Proposition 2.9.

Proposition 2.9 *Let u be a non negative, radial continuous supersolution of*

$$F(x, \nabla u, D^2u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x)u^{1+\alpha} \leq 0,$$

with $c(x) \geq 0$. Then there exists $\delta > 0$ and $K_\delta > 0$ such that, $\forall r \in]1 - \delta, 1[$:

$$|D_l u(r)| \geq K_\delta |D_l u(1)|, \quad |d_l u(r)| \geq K_\delta |d_l u(1)|.$$

Remark 2.10 *Of course, if $\Omega = B(0, 1) \setminus \overline{B(0, \rho)}$, one has the symmetric result near the point $r = \rho$: There exists $\delta > 0$ and $K_\delta > 0$ such that for all $r \in]\rho, \rho + \delta[$*

$$|d_r u(r)| \geq K_\delta |d_r(\rho)| \quad \text{and} \quad |D_r u(r)| \geq K_\delta |D_r(\rho)|.$$

Proof. Let r_u be such that $u(r_u) = \sup u$.

Claim *For all $r_o \in]r_u, 1[$, for all $\delta_1 < r_o - r_u$ and for all $r \in]r_o - \delta_1, r_o[$, and defining $\sigma = \frac{2A(N-1)}{a(r_o - \delta_1)}$,*

$$u(r) \geq u(r_o) + \frac{u(r_o - \delta_1) - u(r_o)}{e^{-\sigma(r_o - \delta_1)} - e^{-\sigma r_o}} (e^{-\sigma r} - e^{-\sigma r_o}).$$

Proof of the claim. It is enough to remark that, with the above choice of σ , the function

$$v(r) = u(r_o) + C(e^{-\sigma r} - e^{-\sigma r_o})$$

is a strict-subsolution for any $C > 0$. Moreover, with the choice of C as in the claim, $v(r_o) = u(r_o)$ and $v(r_o - \delta_1) = u(r_o - \delta_1)$. Hence to conclude the proof of the claim it is enough to apply the comparison theorem.

Using the fundamental calculus theorem, we deduce from the claim that,

$$d_l u(r_o) \leq e^{-\sigma \delta_1} \frac{u(r_o - \delta_1) - u(r_o)}{-\delta_1}. \quad (2.3)$$

Second step : Let $\delta_1 < \frac{1-r_u}{2}$ be such that $\frac{u(1-\delta_1)-u(1)}{\delta_1} \geq \frac{3|d_l u(1)|}{4}$; by continuity, there exists $\delta < \frac{\delta_1}{2}$, such that for all $r \in]1 - \delta, 1[$

$$\left| \frac{u(r - \delta_1) - u(r)}{\delta_1} \right| \geq \frac{|d_l u(1)|}{2}.$$

This, together with (2.3), gives the result:

$$d_l u(r) \leq -e^{-\sigma \delta_1} \frac{|d_l u(1)|}{2} < 0.$$

Similarly for D_l instead of d_l .

Corollary 2.11 *Under the assumptions of Proposition 2.9, $\exists \delta > 0$ such that for all $r \in]1 - \delta, 1[$*

$$\sup(d_l u(r), d_r u(r)) \leq K_\delta \sup(d_l u(1), D_l u(1)) < 0.$$

Proof We use the fact that $D_l u(r) \geq d_r u(r)$, so using proposition 2.9

$$d_r u(r) \leq D_l u(r) \leq K_\delta D_l u(1) < 0$$

and then

$$\sup(d_r u(r), d_l u(r)) \leq \sup(K_\delta d_l u(1), K_\delta D_l u(1)) < 0.$$

Remark 2.12 *The same reasoning establishes that for all $\bar{r} \in]\rho, r_u[$, there exists $\delta > 0$ and $K_{\bar{r}} > 0$ such that for $r \in]\rho - \delta, \rho[$,*

$$|d_l u(r)| \geq K_{\bar{r}} |d_l u(\bar{r})|$$

$$|D_l u(r)| \geq K_{\bar{r}} |D_l u(\bar{r})|$$

Remark 2.13 *As suggested by one of the referee, let us note that this implies that for all $\bar{r} \in]\rho, r_u[$, there exists $\delta > 0$ and $K_{\bar{r}} > 0$ such that in the distributional sense on $] \rho - \delta, \rho[$, $u' \geq K_{\bar{r}} |d_l u(\bar{r})|$. Indeed let $\varphi \in \mathcal{D}'(] \rho - \delta, \rho[$ ≥ 0*

$$\begin{aligned} \int u' \varphi &= - \int u \varphi' \\ &= - \lim_{h \rightarrow 0, h > 0} \int u \frac{\varphi(r+h) - \varphi(r)}{h} dr \\ &= \lim_{h \rightarrow 0, h > 0} \int \frac{u(r-h) - u(r)}{h} \varphi(r) dr \\ &\geq K_{\bar{r}} |d_l u(\bar{r})| \int \varphi(r) dr \end{aligned}$$

3 Uniqueness

We are now in a position to prove the simplicity of the radial eigenfunctions:

Theorem 3.1 *Suppose that F satisfies (H1), (H2), (H3) and (H4), that $c(x) + \bar{\lambda} > 0$ in Ω , and that there exist two positive eigenfunctions ϕ and ψ which are radial, then there exists a constant t such that $\phi = t\psi$.*

Before starting the proof we shall give a few propositions that will be used in the proof of the theorem and which are of independent interest.

The first two results are not specific to radial solutions.

Proposition 3.2 *Suppose that $0 \leq \lambda < \bar{\lambda}(\Omega)$, that $c + \lambda$ is positive on $\bar{\Omega}$ and that u and v are respectively positive continuous super and sub solutions of*

$$F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + (c + \lambda)u^{1+\alpha} \leq 0$$

$$F(x, \nabla v, D^2 v) + b(x) \cdot \nabla v |\nabla v|^\alpha + (c + \lambda)v^{1+\alpha} \geq 0.$$

- 1) *If $u \geq v > 0$ on $\partial\Omega$ then $u \geq v$ in Ω .*
- 2) *If $u > v$ on $\partial\Omega$ then $u > v$ on $\bar{\Omega}$.*

In [6] we have obtained a comparison principle for strict sub and super solutions (Theorem 1), we shall use it here and in the following.

Proof of Proposition 3.2.

In the first case, let $0 < \epsilon < \frac{\min u}{2}$, since by hypothesis u is strictly positive and continuous. Then, since $\lambda + c > 0$ in $\bar{\Omega}$, $u_\epsilon = u - \epsilon \geq 0$ satisfies for some constant $m > 0$

$$F(x, \nabla u_\epsilon, D^2 u_\epsilon) + b(x) \cdot \nabla u_\epsilon |\nabla u_\epsilon|^\alpha + (\lambda + c)(u_\epsilon)^{1+\alpha} \leq -m < 0.$$

While for γ_ϵ defined as $\gamma(\epsilon) = \frac{\epsilon}{\min u - \epsilon}$, $v_\epsilon = \frac{v}{1 + \gamma_\epsilon}$ satisfies

$$F(x, \nabla v_\epsilon, D^2 v_\epsilon) + b(x) \cdot \nabla v_\epsilon |\nabla v_\epsilon|^\alpha + (\lambda + c)v_\epsilon^{1+\alpha} \geq 0.$$

For this choice of $\gamma(\epsilon)$, $u_\epsilon \geq v_\epsilon$ on the boundary. Using the comparison principle in [6], one gets that $u_\epsilon \geq v_\epsilon$ and letting ϵ tend to zero one gets that $u \geq v$.

In the second case, i.e. $u > v$ on the boundary, $u_\epsilon \geq (1 + \epsilon)v$ for some $\epsilon > 0$ and since $v(1 + \epsilon)$ is still a solution, one gets that $u \geq (1 + \epsilon)v$ in Ω .

In the rest of the paper we shall suppose that F satisfies (H1), (H2), (H3), (H4), that b and c are continuous and b satisfies (H5).

Proposition 3.3 *Let $\Omega' \subset\subset \Omega$, suppose that $c(x) + \bar{\lambda}(\Omega) > 0$ then*

$$\bar{\lambda}(\Omega) < \bar{\lambda}(\Omega')$$

Proof. Let ϕ be a positive eigenfunction for $\bar{\lambda}(\Omega)$. Since ϕ is continuous, let $0 < \varepsilon \leq \inf_{\Omega'} \phi$. Then there exists $\lambda' > \bar{\lambda}(\Omega)$ such that $\phi_\varepsilon = \phi - \varepsilon$ is a positive solution of

$$F(x, \nabla \phi_\varepsilon, D^2 \phi_\varepsilon) + b(x) \cdot \nabla \phi_\varepsilon |\nabla \phi_\varepsilon|^\alpha + (\lambda' + c)(\phi_\varepsilon)^{1+\alpha} \leq 0.$$

Indeed just choose λ' such that $\frac{c+\lambda'}{c+\bar{\lambda}(\Omega)} \leq \left(\frac{\sup_{\Omega'} \phi}{\sup_{\Omega'} (\phi - \varepsilon)} \right)^{1+\alpha}$. By definition $\bar{\lambda}(\Omega') \geq \lambda' > \bar{\lambda}(\Omega)$.

The next Proposition is a sort of interior Hopf principle:

Proposition 3.4 *Suppose that $c \geq 0$. Suppose that u and v are respectively nonnegative radial continuous solutions of*

$$F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x)u^{\alpha+1} \leq f$$

and

$$g \leq F(x, \nabla v, D^2 v) + b(x) \cdot \nabla v |\nabla v|^\alpha + c(x)v^{\alpha+1}$$

with $f \leq g$. Suppose that $u \geq v$ in $] \bar{r} - \delta, \bar{r} + \delta[\cap \Omega$ for some $\bar{r} \in [\rho, 1]$, and that $u(\bar{r}) = v(\bar{r})$.

Suppose that $r_1 \in] \bar{r} - \delta, \bar{r} \cap \Omega$ is such that $u(r_1) > v(r_1)$, suppose that either $u' \not\equiv 0$ on $] \bar{r} - \delta, \bar{r} + \delta[\cap \Omega$ or $v' \not\equiv 0$ on $] \bar{r} - \delta, \bar{r} + \delta[\cap \Omega$. Then there exists $C > 0$ which depends only on r_1 and on the data, such that

$$u(r) \geq v(r) + C(\bar{r} - r)$$

for any $r \in]r_1, \bar{r}[$.

Remark 3.5 *Symmetric result holds on the right of \bar{r} : Suppose that $u \geq v$ on a neighborhood $] \bar{r} - \delta, \bar{r} + \delta[\cap \Omega$ of \bar{r} and that $u(\bar{r}) = v(\bar{r})$.*

Suppose that $r_1 > \bar{r}$ is such that $u(r_1) > v(r_1)$, suppose that $u' \not\equiv 0$ on $] \bar{r} - \delta, \bar{r} + \delta[$. Then there exists $C > 0$ which depends only on r_1 and of the data, which is such that for $r \in] \bar{r}, r_1[$

$$u(r) \geq v(r) + C(r - \bar{r}).$$

Remark 3.6 We shall apply this proposition to prove uniqueness, by using the fact that since u is a supersolution, $u'(r) \not\approx 0$ for $r \in]\rho, r_u[$ and $r \in]r_u, 1[$.

Proof of Proposition 3.4 It is sufficient to prove the result when $c = 0$.

Indeed suppose that it has been proved in this case. We get the result using the one obtained in the case $c = 0$, replacing f by $f - c(x)u^{1+\alpha}$, and g by $g - c(x)v^{1+\alpha}$ which also satisfy $f - c(x)u^{1+\alpha} \leq g - c(x)v^{1+\alpha}$. Hence we now suppose that $c = 0$.

Let \bar{r} be in $] \rho, 1[$ such that $u(\bar{r}) = v(\bar{r})$ and $u \geq v$ in $B(0, 1) \setminus \overline{B(0, \rho)}$.

Suppose to fix the ideas that $v'(\bar{r}) \not\approx 0$, then by Definition 2.5, Proposition 2.9 and Proposition 2.6 there exists $k > 0$ and a neighborhood of \bar{r} on which the test functions φ of v satisfy $|\varphi'| \geq k$.

On the other hand, by Lipschitz regularity, we know that there exists L such that $|\varphi'| \leq L$.

Let $\sigma = \sup \left\{ \frac{2A(N-1)}{a(\bar{r}-\delta)}, \left(\frac{2^{|\alpha|+2}|g|_\infty}{ak} + \frac{2^{|\alpha|+1}|h|_\infty}{a} \right) \right\}$ and $\epsilon = \inf \left\{ \frac{ke^{-\sigma\delta}}{2\sigma}, \frac{u(\bar{r}-\delta)-v(\bar{r}-\delta)}{2} \right\}$.

Let $w = e^{\sigma(\bar{r}-r)} - 1$. We shall prove that $\psi(r) = v(r) + \epsilon w(r)$ is a radial sub-solution of

$$F(x, \nabla\psi, D^2\psi) + b(x) \cdot \nabla\psi |\nabla\psi|^\alpha \geq g + m\epsilon$$

for some constant $m > 0$ in the set $[\bar{r} - \delta, \bar{r}]$.

(In the case where $u'(\bar{r}) \not\approx 0$, one would prove in the same manner that $u - \epsilon w$ is a supersolution of

$$F(x, \nabla(u-\epsilon w), D^2(u-\epsilon w)) + b(x) \cdot \nabla(u-\epsilon w) |\nabla(u-\epsilon w)|^\alpha + c|u-\epsilon w|^\alpha(u-\epsilon w) \leq f - m\epsilon$$

for some positive constant m .)

Observe that

$$\varphi' + \epsilon w' = \varphi' - \epsilon \sigma e^{\sigma(\bar{r}-r)}$$

and, with our choice of ϵ , $|\varphi' + \epsilon w'| \geq \frac{|\varphi'|}{2}$.

We then obtain, by (H2),

$$\begin{aligned} & F(x, \nabla(\varphi + \epsilon w), D^2(\varphi + \epsilon w)) + b(x) \cdot \nabla(\varphi + \epsilon w) |\nabla(\varphi + \epsilon w)|^\alpha \\ & \geq |(\varphi' + \epsilon w')|^\alpha \left(\tilde{F}(r, 1, \varphi'') + h(r)\varphi' + \right. \\ & \quad \left. + \epsilon \mathcal{M}_{a,A}^-(D^2w) + \epsilon h(r)w' \right) \\ & \geq |(\varphi' + \epsilon w')|^\alpha \left(\frac{g(x)}{|\varphi'|^\alpha} + \epsilon \mathcal{M}_{a,A}^-(D^2w) + \epsilon h(r)w' \right). \end{aligned}$$

Clearly:

$$\mathcal{M}_{a,A}^-(D^2w) = e^{\sigma(\bar{r}-r)} \left[a\sigma^2 - \frac{A\sigma(n-1)}{r} \right] \geq e^{\sigma(\bar{r}-r)} \frac{a\sigma^2}{2}$$

and

$$|(\varphi' + \epsilon w')|^\alpha \left(\frac{g}{|\varphi'|^\alpha} \right) \geq g - \epsilon \sigma e^{\sigma(\bar{r}-r)} |\alpha| |g|_\infty \frac{2^{|\alpha-1|}}{k}.$$

Putting everything together

$$\begin{aligned} & F(x, \nabla(\varphi + \epsilon w), D^2(\varphi + \epsilon w)) + b(x) \cdot \nabla(\varphi + \epsilon w) |\nabla(\varphi + \epsilon w)|^\alpha \\ & \geq g(x) + |\varphi'|^\alpha \epsilon e^{\sigma(\bar{r}-r)} \left[\frac{a\sigma^2}{2} - \sigma |\alpha| |g|_\infty \frac{2^{|\alpha-1|}}{k} - \sigma 2^{|\alpha|} |h|_\infty \right]. \end{aligned}$$

Which is the required result with our choice of σ .

On the other hand $u(\bar{r}) = v(\bar{r}) = (v + \epsilon w)(\bar{r})$ while, with our choice of ϵ , $u(\bar{r} - \delta) > (v + \epsilon w)(\bar{r} - \delta)$.

Using the comparison principle in the annulus $]\bar{r} - \delta, \bar{r}[$ one gets that $u \geq v + \epsilon w$ in that set and we also get that for $\bar{r} - \delta < r < \bar{r}$

$$u(r) \geq v(r) + \epsilon \sigma (\bar{r} - r).$$

(In particular $d_l u(\bar{r}) \leq d_l v(\bar{r}) - \sigma \epsilon$, $D_l u(\bar{r}) \leq D_l v(\bar{r}) - \sigma \epsilon$ with σ and ϵ some positive constants which depend only on \bar{r} , δ , g , h , a , A).

One can do the same on the right hand side of \bar{r} , more precisely one defines

$$w = e^{\sigma(r-\bar{r})} - 1,$$

and choosing σ large enough, and ϵ small enough, we obtain that $u > v + \epsilon w$ for some ϵ .

Proof of Theorem 3.1 Let us recall that either $\Omega = B(0, 1)$ or $\Omega = B(0, 1) \setminus \overline{B(0, \rho)}$. We shall give the proof in this second case, the other proof being similar but easier.

Let $\Gamma = \sup \frac{\psi}{\phi}$ and $\gamma = \inf \frac{\psi}{\phi}$. We know that by Proposition 3.4 in [6] and the Hopf principle, $\Gamma < \infty$ and $\gamma > 0$. We want to prove that $\Gamma = \gamma$.

Step 1 The extrema of $\frac{\psi}{\phi}$ are reached on the boundary, in the following sense : There exists some sequence r_n which goes either to $r = 1$ or to $r = \rho$ with $\frac{\psi}{\phi}(r_n) \rightarrow \Gamma$, and there exists r'_n which goes either to $r = 1$ or to $r = \rho$ with $\frac{\psi}{\phi}(r'_n) \rightarrow \gamma$.

To prove this claim, let K_n be a strictly increasing sequence of annulus of center 0, $\cup K_n = \Omega$, such that $\overline{K_n} \subset\subset \Omega$. Then $\bar{\lambda}(K_n) > \bar{\lambda}(\Omega)$ by proposition 3.3.

We prove that $\sup_{\Omega \setminus \overline{K_n}} \frac{\psi}{\phi} \rightarrow \Gamma$.

Assume by contradiction that $\lim_{n \rightarrow +\infty} \sup_{\Omega \setminus \overline{K_n}} \frac{\psi}{\phi} < \Gamma$.

Let $\delta > 0$ be such that $\Gamma - \delta \geq \gamma$, and such that, for $n \geq N$, $\sup_{\Omega \setminus \overline{K_n}} \frac{\psi}{\phi} \leq \Gamma - \delta$, one would have on ∂K_n

$$\frac{\psi}{\phi} \leq \Gamma - \delta$$

and using the comparison principle in Proposition 3.2 on K_n , one would get that $\frac{\psi}{\phi} \leq \Gamma - \delta$ both in $\Omega \setminus K_n$ and in K_n for $n \geq N$, which contradicts the definition of Γ .

A similar proof will imply that $\inf_{\Omega \setminus \overline{K_n}} \frac{\psi}{\phi} \rightarrow \gamma$.

Step 2 We prove that $\frac{\psi}{\phi}$ admits a limit on each of the two parts of the boundary, these two limits being respectively the supremum and the infimum of the ratio. Without loss of generality we can suppose that the supremum Γ is reached at $r = 1$

In what follows, we shall prove that $\frac{\psi}{\phi}(r)$ converges to Γ when r goes to 1. Once this will be done, we shall derive similarly that, $\frac{\psi}{\phi}(r)$ converges to γ when r goes to ρ .

Let r_n be a strictly increasing sequence such that $\frac{\psi}{\phi}(r_n) = \Gamma_n \rightarrow \Gamma$. Let $\delta > 0$ and N_0 be such that for $n \geq N_0$, $\frac{\psi}{\phi}(r_n) \geq \Gamma - \delta$. Let then $\delta_1 = 1 - r_{N_0}$. We now prove that for any $r \in]1 - \delta_1, 1[$, $\frac{\psi}{\phi}(r) \geq \Gamma - \delta$.

Indeed for any $n \geq N_0$, since the inequality holds on the boundary of $B(0, r_n) \setminus B(0, r_{n+1})$ using Proposition 3.2, we obtain that $\frac{\psi}{\phi} \geq \Gamma - \delta$ in $B(0, r_n) \setminus B(0, r_{n+1})$. Since $B(0, 1) \setminus B(0, r_{N_0}) = \cup_{n \geq N_0} B(0, r_n) \setminus B(0, r_{n+1})$, we have obtained the required result.

Final step. We prove that $\psi \equiv \Gamma\phi$ in a left neighborhood of 1. Suppose not then there exists δ such that $\delta < \frac{1-r_n}{2}$, with $\psi(1 - \delta) < \Gamma\phi(1 - \delta)$ and

$$\psi \leq \Gamma\phi$$

in a left neighborhood of 1. By Proposition 3.4, there exists $C > 0$ which depends only on δ and on the data, such that for $r \in [1 - \delta, 1[$

$$\psi(r) \leq \Gamma(\phi(r) + C(r - 1)). \tag{3.1}$$

But ϕ is Lipschitz with Lipschitz constant L , hence

$$\frac{1-r}{\phi(r)} \geq \frac{1}{L}.$$

Dividing (3.1) by ϕ and passing to the limit when r goes to 1 we get

$$\Gamma \leq \Gamma \left(1 - \frac{C}{L}\right),$$

a contradiction. We would do the same for $r = \rho$.

Let $\bar{r} = \inf\{r > r_u, \text{ such that } \psi = \Gamma\phi, \text{ in } (r, 1]\}$. We shall prove that $\bar{r} = r_u$.

If $\bar{r} > r_u$, then $\psi'(\bar{r}) \not\sim 0$. This will imply, using the above reasoning that $\frac{\psi}{\phi} = \Gamma$ on a neighborhood $[\bar{r} - \delta, \bar{r}]$ for $\delta < \bar{r} - r_u$. The only differences in the arguments consist in observing that

$$\lim_{r \rightarrow \bar{r}^-} \frac{\psi(r) - \psi(\bar{r})}{\phi(r) - \phi(\bar{r})} = \Gamma.$$

Using the fact that if for some $\delta > 0$, $\psi(\bar{r} - \delta) > \Gamma\phi(\bar{r} - \delta)$ then for some constant $C > 0$

$$\psi \leq \Gamma(\phi + C(r - \bar{r})).$$

Now since ϕ is Lipschitz:

$$\lim_{r \rightarrow \bar{r}^-} \frac{\psi(r) - \psi(\bar{r})}{\phi(r) - \phi(\bar{r})} \leq \Gamma \left(1 - \frac{C}{L}\right).$$

A contradiction hence $\bar{r} = r_u$.

Finally $\frac{\psi}{\phi} = \Gamma$ for $r \in]r_u, 1[$ and similarly we get $\frac{\psi}{\phi} = \gamma$ on $]\rho, r_u[$ a contradiction with the continuity of the functions if $\Gamma \neq \gamma$.

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