Overdetermined problems for fully non linear operators.

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Abstract

In this paper, we consider the equation $|\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u) = -f(u)$ in a bounded smooth domain Ω, with both Dirichlet condition $u = 0$ and Neumann condition $\partial_{\vec{n}} u = c$ on $\partial\Omega$, where c is a constant, $\alpha > -1$, u is of constant sign and $\mathcal{M}_{a,A}$ is one of the Pucci's operator. We prove, for different nonlinearities f, that, when a is sufficiently close to A, either $u = c = 0 = f(0)$ or Ω is a ball, u is radial, and $c \neq 0$ in Ω.

1 Introduction

In this paper, for a large class of nonlinearities $f(u)$, for $\mathcal{M}_{a,A}$ one of the Pucci operators (i.e. either $\mathcal{M}_{a,A} = \mathcal{M}^+_{a,A}$ or $\mathcal{M}_{a,A} = \mathcal{M}^-_{a,A}$) and $\alpha > -1$, we prove that if Ω is a bounded smooth domain, such that there exists u a viscosity, constant sign $\mathcal{C}^1$ solution of

$$\begin{cases} |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_{\vec{n}} u = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for some constant c, then

$$\text{either} \quad c = f(0) = 0 \equiv u \quad \text{or} \quad \Omega \quad \text{is a ball and u is radial.}$$

Here and in the whole paper $\vec{n}$ or $\vec{n}_\Omega$ denotes the unit outer normal to Ω.

Overdetermined boundary value problem is a very rich field, somehow started by Serrin in the acclaimed paper [23] where it is proved that, if u is a solution of

$$\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \vec{n}} = c & \text{on } \partial\Omega, \end{cases}$$

then Ω is a ball and u is radial.

Serrin's proof relies on the method of moving planes. Let us remark that this method has already been extended to prove symmetry of solutions for fully nonlinear equations both by Gidas, Ni, Nirenberg [15] and by Da Lio, Sirakov [10].

On the other hand the overdetermined problem has been greatly generalized to all kind of settings and geometries and it would be far too long to enumerate all the interesting results achieved. We shall only recall the papers by Farina and Kawohl [13] and Buttazzo, Kawohl [8] who consider quasi-linear operators, namely generalization of the p-Laplace operator and the ∞ laplacian. Beside [8], all these results concern divergence form operators. Let us also mention a recent paper of Farina and Valdinoci [14] which treats the case of partially overdetermined problems, i.e. for which the condition $\partial_n u$ is prescribed only on one part of the boundary.

We now want to motivate the results obtained here, hence we shall describe an interesting connection with principal eigenvalues.

Precisely, let $\lambda(\Omega)$ be the functional that associates to a domain Ω the principal eigenvalue of the Dirichlet problem for the Laplace operator. As it is well explained in [11], a domain Ω is critical for the first eigenvalue functional under fixed volume variation if and only if the eigenfunction $\phi > 0$ associated to $\lambda(\Omega)$ has constant Neumann boundary condition i.e. it is a solution of an overdetermined problem. This is proved using the famous Hadamard equality (we refer to [11] and references therein). In [21], Pacard and Sicbaldi have extended this result to Riemann manifolds.

In recent years, the concept of principal eigenvalue has been extended to fully nonlinear operators, by means of the maximum principle (see [2]). The values

$$\lambda^+(\Omega) = \sup\{\lambda, \exists \phi > 0 \text{ in } \Omega, |\nabla\phi|^\alpha \mathcal{M}_{a,A}(D^2\phi) + \lambda\phi^{1+\alpha} \leq 0 \quad \text{in } \Omega\}$$

$$\lambda^-(\Omega) = \sup\{\lambda, \exists \psi < 0 \text{ in } \Omega, |\nabla\psi|^\alpha \mathcal{M}_{a,A}(D^2\psi) + \lambda|\psi|^\alpha\psi \geq 0 \quad \text{in } \Omega\}$$

are generalized eigenvalues in the sense that there exists a non trivial solution to the Dirichlet problem

$$|\nabla\phi|^\alpha \mathcal{M}_{a,A}(D^2\phi) + \lambda^\pm(\Omega)|\phi|^\alpha\phi = 0 \quad \text{in } \ \Omega, \ \phi = 0 \quad \text{on} \quad \partial\Omega.$$

One of the open questions, even for the Pucci operator, is whether the Faber-Krahn inequality holds in this context i.e. suppose that Ω is a domain of volume V and suppose that B is a ball with the same volume, is it true that

$$\lambda^+(B) \leq \lambda^+(\Omega)?$$

A first step in this direction is to prove that the ball is critical for $\lambda^+(\Omega)$ under fixed volume variation. In view of what was described above for the Laplacian, the result obtained here i.e. that the only bounded domain for which the eigenfunction has constant boundary data is the ball, gives a good evidence that it may be the case that the ball is the only critical domain.

For unbounded domains the situation is slightly different, in [24], B. Sirakov considers the case of exterior domains and domains with several connected components and in this reference he also proves that in order to have an overdetermined solution the domain has to be radial. Recently, in dimension 2, Helein, Hauswirth, and Pacard in [16] have constructed a non bounded domain for which there exists a harmonic function with zero Dirichlet data and constant Neumann boundary value, which is neither radial nor an exterior domain. The construction of this domain is deeply related to the Laplace operator, but it would be interesting to know if a similar counterexample can be found for the Pucci operator. This could be the object of a future work.

We come now to a better description of the results contained in this note. It is well known that the last step in Serrin's proof is a sort of Hopf's lemma in "corners". Indeed, if the domain contains a squared corner, and two different and ordered solutions touch each other at this corner, then, for any direction entering the domain, if the derivatives coincide then the second derivatives have to be separated. Interestingly, this result is a consequence of the fact that the eigenvalue of the Laplace Beltrami operator in a quarter sphere S^{N-1} is exactly $2N$, even though this is not obvious at all from Serrin's proof. In Proposition 4.1 we extend Serrin's result to the nonlinear setting considered here as long as a is close to A. Here the difficulty is both that one needs to introduce a generalization of the Pucci's operator on the sphere and to estimate the eigenvalue on the quarter sphere; furthermore it is possible to prove that this eigenvalue is greater than $2Na$. This is where we are led to choose a close to A. Let us point out that, in the case $\alpha = 0$, the barrier function constructed in order to prove Proposition 4.1, is related to some results in the recent preprint [1]. In that preprint S. Armstrong, B. Sirakov and Ch. Smart consider more general operators that are uniformly elliptic and these barrier functions are constructed for other purposes.

The paper is organized in the following way. In the next section we state the results concerning the overdetermined problem, in the third section after recalling known results we prove a comparison principle which is new and interesting in itself, the last section is devoted to the proofs of the main result including the "Hopf lemma in corner" described above.

2 The main result

In the whole paper, for some $h \in (0,1)$, Ω is a bounded $\mathcal{C}^{2,h}$ domain of $\mathbb{R}^N$, $\alpha > -1$, and F is defined by

$$F(p,X) := |p|^\alpha \mathcal{M}_{a,A}(X)$$

where either $\mathcal{M}_{a,A} = \mathcal{M}^+_{a,A}(X) = Atr(X^+) - atr(X^-)$ or $\mathcal{M}_{a,A} = \mathcal{M}^-_{a,A}(X) = atr(X^+) - Atr(X^-)$.

We now state our main result for non negative solutions of (2.1), a symmetric result can be stated for non positive solutions.

Theorem 2.1 *Let f be an Hölder continuous function such that one of these holds:*

(1) $f : [0,+\infty) \to \mathbb{R}$, $f^-(u) \le k u^{\max\{1,1+\alpha\}}$ *for some* $k > 0$, f *is non increasing and positive or* f *is decreasing.*

(2) $f(u) = \lambda u^{1+\alpha} - k u^{1+\beta}$ *with* $\beta > \alpha$, $\lambda > 0$ *and* $k \ge 0$.

(3) $\alpha = 0$, f *is locally Lipschitz continuous.*

There exists a constant $t_1 = t_1(a,f) > 1$ *such that for* $t < t_1$, *if, for* $A = ta$, *there exists* u *a non negative* $\mathcal{C}^1$ *viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_{\vec{n}} u = c & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where c *is a constant, then*

$$\textit{either} \quad c = f(0) = 0 \equiv u \quad \textit{or} \quad \Omega \textit{ is a ball and } u \textit{ is radial.}$$

As explained in the introduction, we are mostly interested in the case $f(u) = \lambda u^{1+\alpha}$. When $\alpha \neq 0$, the conditions on f are dictated by the fact that the proof of Theorem 2.1 uses a Hopf Lemma (Lemma 3.9 below), a strict comparison principle (Corollary 3.11) and comparison Theorems 3.5 or 3.7. Of course there maybe other nonlinearities for which these theorems hold and in alternative one could consider any f that satisfies the hypothesis mentioned in the result above.

Remark 2.2 *For $\alpha \leq 0$ the $\mathcal{C}^1$ regularity of the solution is a consequence of the results in [5, 9]. When $\alpha > 0$ this regularity is an open question, except in the radial case [7] , in the one dimensional case or for operators in divergence form. While we were finishing this paper we received a preprint of Imbert and Silvestre [17] where the authors prove the interior $\mathcal{C}^{1,\gamma}$ regularity in the case $\alpha > 0$.*

Remark 2.3 *As an example for $f \equiv 1$ and for $\mathcal{M}_{a,A} = \mathcal{M}^+_{a,A}$ the solution is explicite, precisely in $B(0, R_c)$:*

$$\varphi(r) = \frac{\alpha+1}{\alpha+2}\left(\frac{1+\alpha}{a((N-1)(1+\alpha)+1)}\right)^{\frac{1}{1+\alpha}}\left(-r^{\frac{\alpha+2}{\alpha+1}} + R_c^{\frac{\alpha+2}{\alpha+1}}\right)$$

where R_c and c are linked by the relation

$$c = -\left(\frac{1+\alpha}{a((N-1)(1+\alpha)+1)}\right)^{\frac{1}{1+\alpha}} R_c^{\frac{1}{1+\alpha}}.$$

As a consequence of Theorem 2.1, in the case $f(u) = \lambda|u|^\alpha u$, we get

Corollary 2.4 *There exists a constant $t_1 > 1$ such that for $t < t_1$ and $A = ta$, the only bounded smooth domains for which a constant sign eigenfunction has constant normal derivative on the boundary, are balls.*

est ce que c'est assez precis de parler de valeur propre alors que c'est non lineaire, ne faut il pas reecrire l'equation?

3 Preliminary results: comparison principles and regularity.

We begin by recalling the definition of viscosity solution that we adopt in the present context.

Definition 3.1 *$v \in \mathcal{C}(\Omega) \cap L^\infty(\Omega)$ is a viscosity super solution of $F(\nabla v, D^2 v) + f(v) = 0$ if, for all $x_o \in \Omega$,*

-either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = \kappa$ and $f(\kappa) \le 0$,

-or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum on x_0 and $\nabla\varphi(x_0) \neq 0$, one has

$$F(\nabla\varphi(x_0), D^2\varphi(x_0)) + f(v(x_0)) \le 0. \tag{3.1}$$

Of course a symmetric definition can be given for viscosity sub-solutions, and a viscosity solution is a function which is both a super-solution and a sub-solution.

We now recall some classical facts concerning the Pucci's operators.

Proposition 3.2 *[9] Suppose that f is Lipschitz continuous and that u and v are respectively viscosity sub- and supersolutions of*

$$\mathcal{M}_{a,A}(D^2 w) + f(w) = 0 \text{ in } \Omega,$$

and $u \le v$ in Ω.

Then either $u \equiv v$ or $u < v$ in Ω and $\partial_{\vec{n}}(u - v) > 0$ on $\partial\Omega$.

Furthermore a consequence of the famous Alexandrov-Bakelman-Pucci inequality allows to prove a maximum principle in "small domains":

Proposition 3.3 *Given $c(x)$ a bounded function in Ω, there exists δ depending on $|c|_\infty$ and on a, A, and the diameter of Ω, such that for any $\Omega_o \subset \Omega$ satisfying $|\Omega_o| \le \delta$:*

$$\begin{cases} \mathcal{M}_{a,A}(D^2 w) + c(x) w \ge 0 & \text{in } \Omega_o, \\ w \le 0 & \text{on } \partial\Omega_o \end{cases} \Rightarrow w \le 0 \text{ in } \Omega_o.$$

The proof is well known, see [2].

We shall also need the following regularity result :

Proposition 3.4 *[25, 12, 9] Let f be some bounded and Hölder function in Ω. Then for all $a > 0$ there exist $\kappa = \kappa(a, f, \Omega)$, $C = C(a, f, \Omega)$ and $\epsilon > 0$ such that, for all $t \in]1, 1+\epsilon]$, any u viscosity solution of*

$$\begin{cases} \mathcal{M}_{a,ta}(D^2 u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$||u||_{\mathcal{C}^{2,\kappa}(\overline{\Omega})} \le C.$$

In general comparison principles play a key role when one deals with viscosity solutions, here Theorem 3.5 will be used for case 1 and Theorem 3.7 is needed in the case 2.

Theorem 3.5 *[3] Suppose that $\phi > 0$ in Ω and σ are respectively, sub- and super-solutions of*

$$F(\nabla\phi, D^2\phi) + f(\phi) \leq g_1 \text{ in } \Omega,$$
$$F(\nabla\sigma, D^2\sigma) + f(\sigma) \geq g_2 \text{ in } \Omega,$$

with g_1, g_2 and f continuous functions on $\mathbb{R}^+$ such that
-either f is decreasing and $g_1 \leq g_2$,
-or f is non increasing and $g_1 < g_2$.
If $\sigma \leq \phi$ on $\partial\Omega$ then $\sigma \leq \phi$ in Ω.

An immediate consequence which allows, in the main theorem, to cover the case of $f \equiv 1$ as in Serrin's paper, is

Corollary 3.6 *The same conclusion holds if f is non increasing, $f(t) > 0$ and $g_1 \leq 0 \leq g_2$.*

Just observe that for $\phi_\varepsilon = (1+\varepsilon)\phi$,

$$F(\nabla\phi_\varepsilon, D^2\phi_\varepsilon) + f(\phi_\varepsilon) \leq f(\phi_\varepsilon) - (1+\epsilon)^{1+\alpha} f(\phi) < f(\phi_\varepsilon) - f(\phi) \leq 0$$

and, on the other hand,

$$F(\nabla\sigma, D^2\sigma) + f(\sigma) \geq 0.$$

So we are in the hypothesis of Theorem 3.5 with $g_1 = f(\phi_\varepsilon) - (1+\epsilon)^{1+\alpha} f(\phi) < 0 \leq g_2$, and $\phi_\varepsilon \geq \sigma$ on $\partial\Omega$. Letting ϵ go to zero one gets the result.

In order to treat the case 2 in the proof of Theorem 2.1 we shall need the following refined comparison principle, where we have denoted, in a classical way and for simplicity, $F[v] = F(\nabla v, D^2 v)$:

Theorem 3.7 *Assume that $u \geq 0$ and $v \geq 0$ are viscosity solutions of*

$$F[v] + h(v) - g(v) \leq 0 \text{ in } \Omega$$

and

$$F[u] + h(u) - g(u) \geq 0 \text{ in } \Omega,$$

such that $v > 0$ on $\overline{\Omega}$. Here h and g are continuous, positive and non decreasing functions on $\mathbb{R}^+$ such that for some $\beta > \alpha$, for all $s > 1$ and for all $\tau > 0$

- $h(s\tau) \leq s^{1+\alpha} h(\tau)$,
- $g(s\tau) \geq s^{1+\beta} g(\tau) > 0$.

Then the comparison principle holds i.e. if $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.

If $g \equiv 0$ and h is as above and increasing then the same conclusion holds.

The proof is postponed to the end of the section.

Remark 3.8 *In these Theorems, Ω needs not be regular, bounded is sufficient.*

We now state a Hopf lemma and a strong comparison principle that will be needed in the proof of Theorem 2.1.

Lemma 3.9 *Let $u > 0$ such that*

$$F[u] + g(u) \leq 0 \text{ in } \Omega,$$

where g is some continuous function such that on a neighborhood on the right of 0, there exists a constant k such that $g^-(u) \leq k(u)^{1+\alpha}$.

Suppose that $u(P) = 0$ for some $P \in \partial\Omega$ such that Ω satisfies the interior sphere condition in P, then there exists $M > 0$ such that

$$u(x) > M d_{\partial\Omega}(x),$$

where $d_{\partial\Omega}$ is the distance to the boundary.

Proof of Lemma 3.9. Let $P \in \partial\Omega$, and $P_1 \in \Omega$ such that $B(P_1, R) \cap \partial\Omega = \emptyset$, $\overline{B(P_1, R)} \cap \partial\Omega = P$, we define $R = |P - P_1|$ and for $x \in B(P_1, R)$, $r = |x - P_1|$. We can also assume that R is small enough in order that $g^-(u) \leq ku^{1+\alpha}$ on the ball $B(P, R)$. We choose m such that $u \geq m$ on $r = \frac{R}{2}$, and we shall prove that for c large enough, $u \geq m(e^{-cr} - e^{-cR})$ in the annulus $B(P_1, R) \setminus B(P_1, \frac{R}{2})$. This will classically imply the result.

It is sufficient to prove that for $w = m(e^{-cr} - e^{-cR})$ and for c chosen conveniently,

$$F[w] - kw^{1+\alpha} > 0$$

We easily get that $F[w] \geq \frac{ac^{2+\alpha}}{2} m^{1+\alpha} e^{-cr(1+\alpha)}$ as soon as $c > \frac{2A(N-1)}{2aR}$. For $c > \left(\frac{2k}{a}\right)^{\frac{1}{2+\alpha}}$,

$$\begin{aligned} kw^{1+\alpha} &= k(m(e^{-cr} - e^{-cR}))^{1+\alpha} \\ &\leq k(me^{-cr})^{1+\alpha} \\ &< \frac{ac^{2+\alpha}}{2} m^{1+\alpha} e^{-cr(1+\alpha)}. \end{aligned}$$

This ends the proof, using the comparison Theorem 3.5.

We shall also need the following strong comparison principle :

Proposition 3.10 *[5] Let u and v be respectively nonnegative $\mathcal{C}^1(\overline{\Omega})$ viscosity solutions of*

$$F[u] \leq f_1 \ in\ \Omega,$$
$$F[v] \geq f_2 \ in\ \Omega,$$

with $f_1 \leq f_2$ and $u \geq v$ in Ω. In any $\mathcal{O}$ open, connected subset of Ω where ∇u or ∇v does not take the value zero:

$$either\ u \equiv v\ or\ u > v.$$

Furthermore if $v > 0$ in Ω, $v = 0$ on $\partial\Omega$, and if $\bar{x} \in \partial\Omega$ is such that $u(\bar{x}) = 0$, and $\partial_{\vec{n}} u(\bar{x}) = \partial_{\vec{n}} v(\bar{x})$, then there exists $\epsilon > 0$ such that

$$u \equiv v \ in\ \Omega \setminus \overline{\Omega}_\epsilon$$

where Ω_ϵ is the set of points of Ω whose distance to the connected component of the boundary which contains $\bar{x}$ is greater than ϵ.

This proposition holds for a more general class of operators than the one considered here. It will be used in the proof of Theorem 2.1.

Corollary 3.11 *Let u and v be respectively nonnegative $\mathcal{C}^1(\overline{\Omega})$ viscosity solutions of*

$$F[u] + f(u) \leq 0 \ in \quad \Omega,$$
$$F[v] + f(v) \geq 0 \ in \quad \Omega,$$

with $u \geq v$. Let f be some continuous function such that, f is Lipschitz away from 0 and in a neighborhood of 0, $f = f_1 - f_2$ for some nondecreasing functions f_1 and f_2 such that $f_2(u) \leq ku$.

Then the conclusions of Proposition 3.10 hold true.

Proof of Corollary 3.11. We only give a hint of the proof when $P \in \partial\Omega$ and $u(P) = v(P) = 0$ so we need to prove that $u \equiv v$ in a neighborhood of P. Observe that

$$F[u] - f_2(u) \leq -f_1(u) \leq -f_1(v) \leq F[v] - f_2(v).$$

For $f_2 = 0$ this is just Proposition 3.10, otherwise the argument (precisely the test function) used in the proof of Proposition 4.4 in [5] can be extended to this case using the fact that f_2 is Lipschitz near zero 0. The arguments are similar to those in the proof of Proposition 4.1 at the end of the present article.

Proof of Theorem 3.7. We can assume without loss of generality that u and v are positive.

We start by considering the case where $g > 0$. We suppose by contradiction that somewhere $u > v$. Let $\gamma' = \sup_\Omega \frac{u}{v}$, let $\kappa = ((\gamma')^{1+\beta} - (\gamma')^{1+\alpha}) \inf_{x\in\overline{\Omega}} g(v(x))$ and let $\gamma \in]1, \gamma'[$ sufficiently close to γ in order that

$$\sup_{x\in\overline{\Omega}} |h(\gamma v) - h(\gamma' v)| \leq \frac{\kappa}{4}$$

and $(\gamma^{1+\beta} - \gamma^{1+\alpha}) \inf_{x\in\overline{\Omega}} g(v(x)) \geq \dfrac{3\kappa}{4}$. Let us note that $u - \gamma v$ achieves its positive maximum inside Ω.

Let us define $\psi_j(x,y) = u(x) - \gamma v(y) - \frac{j}{q}|x-y|^q$ where $q > \sup(\frac{\alpha+2}{\alpha+1}, 2)$. It is classical that ψ_j achieves its maximum on some pair (x_j, y_j) which is in Ω^2 and that $(x_j, y_j) \to (\bar{x}, \bar{x})$ where

$$u(\bar{x}) - \gamma v(\bar{x}) = \sup_{x\in\overline{\Omega}} (u(x) - \gamma v(x)) > 0.$$

Moreover $j|x_j - y_j|^q \to 0$. Then using Ishii's lemma, see [18, 3] , there exist X_j, Y_j in S with

$$(j|x_j - y_j|^{q-2}(x_j - y_j), X_j) \in J^{2,+}u(x_j),\ (j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \in J^{2,-}v(y_j)$$

with $X_j + \gamma Y_j \leq 0$. If $x_j \neq y_j$ (which will be checked later) from the definition of viscosity solutions

$$\begin{aligned} -h(u(x_j)) + g(u(x_j)) &\leq F(j|x_j - y_j|^{q-2}(x_j - y_j), X_j) \\ &\leq \gamma^{1+\alpha} F(j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \\ &\leq \gamma^{1+\alpha} \left(-h(v(y_j)) + g(v(y_j))\right). \end{aligned}$$

Passing to the limit and using the properties of h and g one obtains

$$\begin{aligned} -h(\gamma' v(\bar{x})) + g(\gamma v(\bar{x})) &\leq -h(u(\bar{x})) + g(u(\bar{x})) \\ &\leq \gamma^{1+\alpha}(-h(v(\bar{x})) + g(v(\bar{x}))) \\ &\leq -h(\gamma v(\bar{x})) + \gamma^{1+\beta} g(v(\bar{x})) - \frac{3\kappa}{4} \\ &\leq -h(\gamma' v(\bar{x})) + g(\gamma v(\bar{x})) - \frac{\kappa}{2}, \end{aligned}$$

which is a contradiction.
We now suppose that $g \equiv 0$ and h is increasing. Suppose first that there exists $\delta > 0$ such that

$$F[v] + h(v) \leq -\delta. \tag{3.2}$$

Since $v > 0$ on $\overline{\Omega}$, we define γ' as before, we want to prove that $\gamma' \leq 1$, then we suppose by contradiction that $\gamma' > 1$. Let $\gamma \in]1, \gamma'[$ be small enough in order that by the continuity of h and the boundedness of v one has

$$\sup_{x \in \overline{\Omega}} |h(\gamma' v(x)) - h(\gamma v(x))| \leq \frac{\delta}{4}.$$

By passing to the limit in (3.2) with $g \equiv 0$, and using the properties of h, we obtain

$$-h(\gamma' v(\bar{x})) \leq -h(u(\bar{x})) \leq -\gamma^{1+\alpha} h(v(\bar{x})) - \delta \leq -h(\gamma' v(\bar{x})) - \frac{\delta}{2},$$

a contradiction.

Suppose (3.2) does not hold, and recall that $v > 0$ on $\overline{\Omega}$. For any arbitrary positive ϵ let $w_\epsilon = v(1+\epsilon) - \frac{\min_\Omega v}{2}\epsilon$. Then $u < w_\epsilon$ on $\partial\Omega$ and since h is now supposed to be increasing, there exists $\delta_\epsilon > 0$ such that $h(w_\epsilon) \leq (1+\epsilon)^{1+\alpha} h(v) - \delta_\epsilon$ hence

$$F[w_\epsilon] + h(w_\epsilon) \leq (1+\epsilon)^{1+\alpha}(F[v] + h(v)) - \delta_\epsilon \leq -\delta_\epsilon$$

and then, from the previous result, $u \leq w_\epsilon$ in Ω, and letting ϵ go to zero, $u \leq v$ in Ω.

There remains to prove that $x_j \neq y_j$ definitely. If $x_j = y_j$, one would have

$$v(x) \geq v(x_j) - \frac{j}{q}|x - x_j|^q \quad \text{and} \quad u(x) \leq u(x_j) + \frac{j}{q}|x - x_j|^q.$$

If the infimum

$$\inf_{x \in B_r(x_j)} \{v(x) + \frac{j}{q}|x - x_j|^q\}$$

is not strict then one can replace x_j by some point y_j close to it and then we are done. The same is true if we assume that the supremum

$$\sup_{x \in B_r(x_j)} \{u(x) - \frac{j}{q}|x - x_j|^q\}$$

is not strict. So we assume that both extrema are strict. In this case, proceeding as in [3] one can prove, using the equation and the definition of viscosity solution, that

$$h(v(x_j)) - g(v(x_j)) \leq 0 \quad \text{and} \quad h(u(x_j)) - g(u(x_j)) \geq 0.$$

Passing to the limit the inequality becomes

$$h(v(\bar{x})) - g(v(\bar{x})) \leq 0 \quad \text{and} \quad h(u(\bar{x})) - g(u(\bar{x})) \geq 0.$$

Using $u(\bar{x}) > v(\bar{x})$ one derives that

$$h(u(\bar{x})) \geq g(u(\bar{x})) \geq \left(\frac{u(\bar{x})}{v(\bar{x})}\right)^{1+\beta} g(v(\bar{x})) \geq \left(\frac{u(\bar{x})}{v(\bar{x})}\right)^{1+\beta} h(v(\bar{x})).$$

Let us note that, by the assumptions on h, since $\left(\frac{u}{v}\right)(\bar{x}) > 1$, one obtains:

$$\left(\left(\frac{u(\bar{x})}{v(\bar{x})}\right)^{1+\alpha} - \left(\frac{u(\bar{x})}{v(\bar{x})}\right)^{1+\beta}\right) h(v(\bar{x})) > 0$$

which is a contradiction for $\beta > \alpha$.

In the case where $g \equiv 0$ the result holds by the increasing behavior of h. This ends the proof of Theorem 3.7.

We end this section with an important remark concerning regularity of solutions close to the boundary :

Remark 3.12 *Observe that, as a consequence of Proposition 3.4, in each of the cases 1,2,3 of Theorem 2.1, we know that for any u, $\mathcal{C}^1$, constant sign solution of*

$$\begin{cases} |\nabla u|^\alpha \, \mathcal{M}_{a,A}(D^2u) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_{\vec{n}} u = c \neq 0 & \text{on } \partial\Omega \end{cases}$$

there exists $\gamma \in (0,1)$ and a neighborhood of $\partial\Omega$ such that $u \in \mathcal{C}^{2,\gamma}$ in that neighborhood.

To prove this regularity in the case $\alpha < 0$, the hypothesis that u is $\mathcal{C}^1$ is not needed, furthermore the result is true everywhere, not only on a neighborhood of the boundary; the proof can be found in [5]. When $\alpha > 0$ one can use the same arguments as in Theorem 2.8 of [6] .

4 Proofs of the main results

As in Serrin's original paper [23] in order to prove Theorem 2.1 we use the moving planes method.

We shall need the two following results :

Proposition 4.1 *Suppose that f is as in Theorem 2.1 . Suppose that Ω is some bounded $\mathcal{C}^{2,h}$ domain, and suppose that H_0 is an hyperplane such that*

- *there exists $P \in H_0 \cap \partial\Omega$ with $\vec{n}_\Omega(P) \in H_0$,*
- *Ω^- is the intersection of Ω with one of the half spaces bounded by H_0 and Ω^+, its reflection with respect to H_o, is contained in Ω.*

Let $u \geq 0$ be a solution of

$$|\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u) + f(u) = 0 \quad in \quad \Omega.$$

Let u_o be the reflected of $u\Big|_{\Omega^-}$ in Ω^+. If $u_o > u$ in a neighborhood of P in Ω, $u(P) = u_o(P) = 0$ and $\nabla u(P) = \nabla u_o(P) \neq 0$, then there exists a $t_1 > 1$ such that for $1 < t < t_1$ and $A = ta$, and for any $\vec{\nu} \in \mathbb{R}^N$ a direction pointing inside Ω^+,

$$\partial^2_{\vec{\nu}} u_o(P) > \partial^2_{\vec{\nu}} u(P).$$

Remark 4.2 *As explained in the introduction, the hypothesis that a is close to A is only needed for the proof of Proposition 4.1.*

Lemma 4.3 *For any $u \in \mathcal{C}^1$ solution of (2.1), if $\partial\Omega$ is the zero level set of a $\mathcal{C}^2$ function ψ, then for any $P \in \partial\Omega$, $D^2u(P)$ depends only on $\psi(P)$, $\nabla\psi(P)$ and $D^2\psi(P)$.*

The proof of Lemma 4.3 proceeds similarly to Serrin's original paper, we include it here for convenience of the reader.

Proof of Lemma 4.3. Observe first that, due to Remark 3.12, close to the boundary the solutions are $\mathcal{C}^2$. Let ϕ be a $\mathcal{C}^2$ function, such that in a neighborhood of P,

$$\psi(x) = x_N - \phi(x_1, \cdots x_{N-1})$$

i.e. $\partial\Omega$ coincides with the graph $x_N = \phi(x_1, \cdots x_{N-1})$. Without loss of generality we can suppose that $P = 0$ and e_N is normal to $\partial\Omega$ in 0, hence $\nabla\phi(0) = 0$. The Neumann boundary condition implies

$$\partial_N u - \sum_{k=1}^{N-1} \partial_k u \partial_k \phi = -c(1 + |\nabla\phi|^2)^{\frac{1}{2}}; \tag{4.1}$$

this, together with the Dirichlet condition differentiated, i.e. for $1 \le i \le N-1$:

$$(\partial_i u + \partial_N u \partial_i \phi)(x_1, \cdots, x_{N-1}, \phi(x_1, \cdots, x_{N-1})) = 0. \tag{4.2}$$

implies $\partial_N u(0) = -c$ and $\partial_i u(0) = 0$.

For $j = 1, \ldots, N-1$, taking the derivative with respect to x_j of (4.2) and (4.1) gives

$$\partial_{ij} u(0) - c\partial_{ij}\phi(0) = 0, \quad \partial_{Nj} u(0) = 0.$$

Finally

$$D^2 u(0) = \begin{pmatrix} cD^2\phi(0) & 0 \\ 0 & \partial_{NN} u(0) \end{pmatrix}.$$

Then, by passing to the limit in the equation one obtains

$$u_{NN}(0) = \beta \left(-\mathcal{M}_{a,A}(cD^2\phi)(0) - |c|^{-\alpha} f(0) \right),$$

here $\mathcal{M}_{a,A}$ is understood as acting on $(N-1) \times (N-1)$ matrices and $\beta = \frac{1}{a}$ or $\frac{1}{A}$ depending on the sign of $-\mathcal{M}_{a,A}(cD^2\phi)(0) - |c|^{-\alpha} f(0)$. This ends the proof of Lemma 4.3.

The proof of Proposition 4.1 is postponed to after the proof of Theorem 2.1. For convenience of the reader we recall the three cases we are going to treat:

(1) $f : [0, +\infty) \to \mathbb{R}$, $f^-(u) \le k u^{\max\{1, 1+\alpha\}}$ for some $k > 0$, f is non increasing and positive or f is decreasing.

(2) $f(u) = \lambda u^{1+\alpha} - k u^{1+\beta}$ with $\beta > 1 + \alpha$, $\lambda > 0$ and $k \ge 0$.

(3) $\alpha = 0$, f is locally Lipschitz continuous.

Proof of Theorem 2.1. We start by remarking that the strong maximum principle holds in all cases (cases 1 and 2 are covered thanks to Lemma 3.9) and hence either $u \equiv 0$ and then $c = f(0) = 0$, or $u > 0$ in Ω and $c < 0$. So we shall suppose that $u > 0$.

In order to start the moving plane procedure, we choose a direction, say e_1, and for $t \in \mathbb{R}$, we denote by H_t the hyperplane $\{x_1 = t\}$ and the sets $\Omega_t^- = \Omega \cap \{x_1 < t\}$, and $\Omega_t^+ = \{x, (2t - x_1, x') \in \Omega_t^-\}$.

We define $u_t(x) = u(2t - x_1, x')$. It is easy to see that for any $\phi \in \mathcal{C}^2$, the eigenvalues of the Hessian of ϕ and ϕ_t are the same, as well as the modulus of their gradient. Hence, using the definition of viscosity solution and the definition of Pucci's operator, we get that u and u_t satisfy the same equation in Ω_t^+ and $u \geq u_t$ on $\partial\Omega_t^+$.

It is clear that for $t < 0$ large, $\Omega_t^- = \emptyset$. Let $t_1 = \sup\{t, \ \forall s < t \ , \ \Omega_s^- = \emptyset\}$ and $t^\star = \sup\{\tilde{t}, \ \forall t < \tilde{t} \ , \ \Omega_t^+ \subset \Omega\}$ then $t^\star$ is such that one of the two following events occurs:

- **event 1** : $H_{t^\star}$ contains the normal to the boundary of Ω at some point P, or

- **event 2** : $\Omega_{t^\star}^+$ becomes internally tangent to the boundary of Ω at some point P not on $H_{t^\star}$.

Recall that for any $t \in (t_1, t^\star)$, $u = u_t$ on H_t, and $u > u_t$ on $\partial\Omega_t^+ \cap \Omega$.

In all cases we need to prove the following two steps:

Step 1 $u_t \leq u$ in Ω_t^+ for any $t \in (t_1, t^\star]$.

Step 2 Ω is symmetric with respect to $H_{t^\star}$ i.e. $\Omega = \Omega_{t^\star}^- \cup \Omega_{t^\star}^+ \cup H_{t^\star}$.

This ends the proof because it implies that Ω and u are symmetric with respect to e_1, but this direction was chosen arbitrarily, so we have obtained that Ω is a ball and u is radial.

Proof of Step 1 in Case 1. It is just an application of Theorem 3.5 in Ω_t^+.

Proof of Step 1 in Case 2. In this case we want to use Theorem 3.7 so we need to restrict to a domain where u is positive away from zero. For $t < t^\star$ there are no points in $\partial\Omega \cap H_t$ with $\vec{n}_\Omega \in H_t$. Then, for all $\bar{x} \in H_t \cap \partial\Omega$, $\vec{n}_\Omega(\bar{x}) \cdot e_1 < 0$ and

$$\partial_{x_1} u(\bar{x}) > 0 \quad \text{and} \quad \partial_{x_1} u_t(\bar{x}) < 0.$$

As a consequence there exists $\epsilon > 0$ such that $u_t \leq u$ in $B(\bar{x}, \epsilon) \cap \Omega_t^+$. Let $B_\epsilon = \cup_{\bar{x} \in \partial\Omega \cap H_t} B(\bar{x}, \epsilon)$.

Let $\Omega_{t,\epsilon} = \Omega_t^+ \setminus \overline{B_\epsilon}$, then $u \geq u_t$ on $\partial\Omega_{t,\epsilon}$, and $u > 0$ on $\overline{\Omega_{t,\epsilon}}$. We are in the hypothesis of Theorem 3.7 hence $u_t \leq u$ in $\Omega_{t,\epsilon}$ and hence in Ω_t^+. By continuity, the inequality holds also for $t = t^\star$.

Proof of Step 1 in Case 3. Let us recall that we are in the case $\alpha = 0$, and that f is only supposed to be Lipschitz continuous. Here the key argument will be the maximum principle in small domains i.e. Proposition 3.3.
We start by proving that, for some $h > 0$, and for $t \in [t_1, t_1 + h[$, $u_t \le u$ in Ω_t^+.

Let $Q \in \partial\Omega \cap H_{t_1}$. Then $\vec{n}_\Omega(Q) = -e_1$, Neumann condition implies that $\partial_{x_1} u(Q) = |c|$, hence since u is $\mathcal{C}^1$, there exists $r > 0$ such that on $B(Q, r) \cap \Omega$, $\partial_{x_1} u(x) \ge \frac{|c|}{2}$. Hence for $|t - t_1|$ small enough and for $t < x_1 < 2t$,

$$u(2t - x_1, x') < u(x_1, x').$$

We now define

$$\bar{t} = \sup\{t \le t^\star, \forall t' < t,\ u_{t'} \le u \text{ in } \Omega_{t'}^+\}.$$

We want to prove that $\bar{t} = t^\star$.

Suppose by contradiction that $\bar{t} < t^\star$, then $\Omega_{\bar{t}+h}^+ \subset \Omega$ for h small enough. Observe that $u_{\bar{t}} < u$ in $\Omega_{\bar{t}}^+$. Indeed, since f is Lipschitz continuous, one can use the strong maximum principle Proposition 3.2 for the difference $u_{\bar{t}} - u$ and obtain both that $u_{\bar{t}} < u$ inside $\Omega_{\bar{t}}^+$ and $\partial_{x_1}(u - u_{\bar{t}}) > 0$ on $\partial\Omega_{\bar{t}} \cap H_{\bar{t}}$.

Claim For $h > 0$ small enough $u_{\bar{t}+h} \le u$ in $\Omega_{\bar{t}+h}^+$.
This claim will contradict the definition of $\bar{t}$.
To prove the claim, let K be a compact subset of $\Omega_{\bar{t}}^+$ such that

$$|\Omega_{\bar{t}}^+ \setminus K| \le 2\delta,$$

where $\delta > 0$ is the constant in Proposition 3.3 that depends on Ω and $|c(x)|_\infty = L_f$ some Lipschitz constant of f around zero. Clearly in K, $u_{\bar{t}} < u$ and, by continuity, for any h sufficiently small, we still have $u_{\bar{t}+h} < u$ in K.
Take h sufficiently small in order that $K \subset\subset \Omega_{\bar{t}+h}^+$ and

$$|\Omega_{\bar{t}+h}^+ \setminus K| \le \delta.$$

Since u and $u_{\bar{t}+h}$ satisfy the same equation in $\Omega_{\bar{t}+h}^+$, $w = u_{\bar{t}+h} - u$ satisfies

$$\mathcal{M}_{a,A}^+ w + L_f w \ge 0 \quad \text{in} \quad \Omega_{\bar{t}+h}^+ \setminus K$$

and $w \le 0$ in $\partial\left(\Omega_{\bar{t}+h}^+ \setminus K\right)$.

By Proposition 3.3, $w \le 0$ in $\Omega_{\bar{t}+h}^+ \setminus K$. Finally $u_{\bar{t}+h} \le u$ in $\Omega_{\bar{t}+h}$, for any $h > 0$ sufficiently small. We have obtained that $\bar{t} = t^\star$.

Proof of step 2 for all cases. First suppose that "event 2" occurs i.e. there exists $P \in \partial\Omega^+_{t^\star} \cap \partial\Omega$. When $\alpha = 0$ the strong maximum principle implies that $u = u_{t^\star}$ in $\Omega^+_{t^\star}$ and Ω is symmetric with respect to $H_{t^\star}$.

For $\alpha \neq 0$, since the unit outer normal to $\partial\Omega$ in P is the same than the one of $\partial\Omega^+_{t^\star}$, by obvious symmetries,

$$\partial_{\vec{n}} u_{t^\star}(P) = \partial_{\vec{n}} u(P) = c.$$

In the case 1, using Corollary 3.11, one gets that $u = u_{t^\star} = 0$ on the connected component of $\partial\Omega^+_{t^\star} \cap \overline{\Omega}$ which contains P. This implies that

$$\partial\Omega \cap \partial\Omega^+_{t^\star} = \partial\Omega^+_{t^\star} \setminus H_{t^\star}$$

i.e. Ω is symmetric with respect to $H_{t^\star}$.

In the case 2, f is increasing only for u small. So in order to apply Corollary 3.11 (with f_2=0) we use the same procedure but considering first a ball around P where u is sufficiently small and then moving with adjacent balls, until one proves that $u = 0$ in the whole boundary $\partial\Omega \cap \partial\Omega^+_{t^\star}$.

We now consider "event 1", i.e. we suppose that there exists some point $P \in H_{t^\star} \cap \partial\Omega$, with $\vec{n}_\Omega(P) \in H_{t^\star}$. We begin to prove that $u = u_{t^\star}$ in a neighborhood of P in Ω_t.

Suppose by contradiction that $u > u_{t^\star}$ inside $\Omega_{t^\star} \cap B(P,R)$, then, by Proposition 4.1, if $\vec{\nu}$ is such that $\vec{\nu} \cdot \vec{n} < 0$, and $\vec{\nu} \cdot \vec{e}_1 > 0$, either $\partial_{\vec{\nu}} u(P) > \partial_{\vec{\nu}} u_{t^\star}(P)$ or $\partial^2_{\vec{\nu}} u(P) > \partial^2_{\vec{\nu}} u_{t^\star}(P)$.

The first inequality is impossible since on $\partial\Omega$, $\partial_{\vec{\nu}} u(P) = c(\vec{\nu} \cdot \vec{n}) = \partial_{\vec{\nu}} u_{t^\star}(P)$. The second inequality is also impossible because Lemma 4.3 implies that $\partial^2_{\vec{\nu}} u(P) = \partial^2_{\vec{\nu}} u_{t^\star}(P)$.

We have obtained that $u = u_{t^\star}$ in a neighborhood of P. This implies in particular that $u = 0$ on $\partial\Omega^+_{t^\star} \cap B(P,R)$ hence by Hopf principle $\partial\Omega^+_{t^\star} \cap B(P,R) \subset \partial\Omega$. Using Corollary 3.11 we get that $u = 0$ in $\partial\Omega^+_{t^\star} \setminus H_{t^\star}$. This of course implies that Ω is symmetric with respect to $H_{t^\star}$.

This ends the proof of step 2 and hence of Theorem 2.1.

The proof of Proposition 4.1 relies on a lemma about barriers by below on the "quarter ball ". For another similar explicit calculation (but with totally different aims) see also [20]. Let us introduce the following map, which is a combination of the spherical coordinates and the stereographic projection:

For $x \in \mathbb{R}^N \setminus \{0\}$, $r = |x|$, $x_N > 0$ and $x' = (x_1, \cdots x_{N-1})$, let $y(x) = \frac{x'}{x_N + r}$.

Lemma 4.4 *For any $\delta > 0$ and any $\gamma > 2$, there exists $t_1 > 1$ and $\epsilon_o > 0$ such that for any $\epsilon \in (0, \epsilon_o)$ and any $t \in (1, t_1)$ there exists $\psi : (\mathbb{R}^{N-1})^+ \to \mathbb{R}$ a $\mathcal{C}^2$ positive function such that $w(x) = r^\gamma \psi(y)$ satisfies*

$$\begin{cases} \mathcal{M}^-_{a,ta}(D^2 w) \geq \epsilon(r^{-2}w + r^{-1}|\nabla w|) & \text{in } \{x \in \mathbb{R}^N,\ |x| \leq R,\ x_N > \delta|x|\} \\ w > 0 & \text{in } \{x_1 > 0, x_N > \delta|x|\}, \\ w = 0 & \text{on } \{x_1 = 0, x_N = \delta|x|\}. \end{cases}$$

We postpone the proof of Lemma 4.4 and prove Proposition 4.1.

Proof of Proposition 4.1.
Without loss of generality we suppose that $H_o = \{x_1 = 0\}$. Let us note first that since $u_o > u$ on a neighborhood of P, $\partial_{\vec{\nu}} u_o(P) \geq \partial_{\vec{\nu}} u(P)$, so we assume that $\partial_{\vec{\nu}} u_o(P) = \partial_{\vec{\nu}} u(P)$, and we want to prove that $\partial^2_{\vec{\nu}} u_o(P) > \partial^2_{\vec{\nu}} u(P)$. One can also assume that $P = 0$. Finally to fix the ideas we take $\alpha \leq 0$, the changes to bring in the inequalities below when $\alpha > 0$ being obvious.

In the lines below l is some positive constant given and we choose m and R depending on it. Later in cases 2 and 3, l will be determined by the condition that $u \mapsto f(u) + lu$ is increasing and in case 1, l can be chosen arbitrary positive.

Let κ given by Proposition 3.4; fixe $\gamma \in (2, 2+\kappa)$ and ϵ_0 and t_1 as in Lemma 4.4. We will prove that there exist $R > 0$ and $m > 0$ such that for $w(x) = r^\gamma \psi(y)$, the function $u + mw$ satisfies

$$\begin{cases} F[u + mw] - l(u + mw) > F[u_o] - lu_o \text{ in } S^+_{\delta,R}, \\ u_o \geq u + mw \text{ on } \partial S^+_{\delta,R} \end{cases} \tag{4.3}$$

with $S^+_{\delta,R} := \{r < R, x_1 > 0,\ x_N > \delta r\} = \{r < R,\ y \in B^+_\delta\}$ and $B^+_\delta = \{y_1 > 0, |y|^2 < \frac{1-\delta}{1+\delta}\}$.

We start by the boundary condition. Observe that $\partial S^+_{\delta,R} = S_1 \cup S_N \cup S_R$ where

$$S_1 = \partial S^+_{\delta,R} \cap \{y_1 = 0\},\ S_N = \partial S^+_{\delta,R} \cap \{y_N = \delta r\},\ S_R = \partial S^+_{\delta,R} \cap B_R(0).$$

By the definition of ψ, $w = 0$ on $S_1 \cup S_N$, so the boundary inequality needs to be checked only on S_R.
Using standard estimates there exists L_ψ such that

$$\psi(y) \leq L_\psi d(y, \partial B^+_\delta).$$

On the other hand, in a neighborhood of $\partial B_\delta^+ \cap B_R(0)$, for some $m' > 0$,

$$(u_o - u)(x) \geq m' d(y, \partial B_\delta^+);$$

this is an application of Corollary 3.11 close to the points where $(u_o - u) = 0$ and it is obvious if we are close to points where $(u_o - u) > 0$.

Putting everything together we obtain that there exists $m > 0$ sufficiently small that

$$(u_o - u)(x) \geq m R^\gamma \psi(y) \text{ for } \; x \in S_R.$$

We now prove (4.3) in $S_{\delta,R}^+$. Since u is $\mathcal{C}^1$, using the Neumann condition at 0, for some $R > 0$,

$$L_2 \geq |\nabla u| \geq L_1 > 0 \text{ in } S_{\delta,R}^+.$$

From Lemma 4.4, using the properties of ψ, there exists $c = c(a, A, N, \gamma) > 0$ such that $|\nabla w| \leq r^{\gamma-1} c$.

We now choose R small enough that

$$cR^{\gamma-1} \leq \inf(\frac{L_1}{2}, \frac{L_2}{2}), \quad \epsilon R^{-2} \left(\frac{3L_2}{2}\right)^\alpha > l \tag{4.4}$$

and

$$\epsilon R^{-1} \left(\frac{3L_2}{2}\right)^\alpha > L_1 2^{-\alpha} |f(u)|_\infty. \tag{4.5}$$

We now observe that by the choice of m and R, $\frac{L_1}{2} \leq |\nabla(u + mw)| \leq \frac{3L_2}{2}$ and then $\left(\frac{3L_2}{2}\right)^\alpha \leq |\nabla(u + mw)|^\alpha \leq \left(\frac{L_1}{2}\right)^\alpha$.

Using (4.4) and (4.5) and Lemma 4.4,

$$\begin{aligned} |\nabla(u + mw)|^\alpha \mathcal{M}_{a,A}(D^2 u + mD^2 w) &\geq |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u) - m|\nabla w|_\infty L_1 2^{-\alpha} |f(u)|_\infty \\ &\quad + \left(\frac{3L_2}{2}\right)^\alpha \mathcal{M}_{a,A}^-(mD^2 w) \\ &\geq |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u) + lmw. \end{aligned}$$

Consequently one has in cases 2 and 3:

$$\begin{aligned} |\nabla(u + mw)|^\alpha \mathcal{M}_{a,A}(D^2 u + mD^2 w) &- l(u + mw) \\ &\geq |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u) - lu \\ &\geq -f(u) - lu \\ &\geq -f(u_o) - lu_o \\ &\geq |\nabla u_o|^\alpha \mathcal{M}_{a,A}(D^2 u_o) - lu_o. \end{aligned}$$

In the third inequality we have used the fact that $u \mapsto f(u) + lu$ is increasing. By Theorem 3.5, this implies that $u_o \geq u + mw$.

We now consider case 1. As we pointed out before here l can be replaced by any constant positive, m is chosen as previously, c is chosen as in (4.4), R is chosen as in (4.5). Hence, similarly to the above inequalities we get:

$$
\begin{aligned}
|\nabla(u+mw)|^\alpha \mathcal{M}_{a,A}(D^2u + mD^2w) &\geq |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u) - m|\nabla w|_\infty L_1 2^{-\alpha}|f(u)|_\infty \\
&\quad + \left(\frac{3L_2}{2}\right)^\alpha \mathcal{M}^-_{a,A}(mD^2w) \\
&\geq |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u) + lmw \\
&\geq -f(u) + lmw \\
&> -f(u_o) + lmw \\
&> |\nabla u_o|^\alpha \mathcal{M}_{a,A}(D^2u_o)
\end{aligned}
$$

This implies, using the first comparison theorem, that $u+mw \leq u_o$ which gives the required result.

When $\alpha > 0$, it is enough to exchange the role of L_1 and L_2 in some of the inequalities above.

We conclude for all the cases. Let $\vec{\nu}$ be any direction pointing inside Ω^+; for δ sufficiently small, $\vec{\nu}$ belongs to the interior of $S^+_{\delta,R}$.

Suppose now that $\partial_{\vec{\nu}} u(0) = \partial_{\vec{\nu}} u_o(0)$, and $\partial^2_{\vec{\nu}} u(0) = \partial^2_{\vec{\nu}} u_o(0)$. By Proposition 3.4, there exists some constant c such that for all $r < R$,

$$(u_o - u)(r\vec{\nu}) \leq cr^{2+\kappa}.$$

This is a contradiction with

$$u_o(r\vec{\nu}) \geq u(r\vec{\nu}) + mw(r\vec{\nu}) = u(r\vec{\nu}) + mr^\gamma \psi(y(\vec{\nu})),$$

since $\gamma < 2 + \kappa$ and ψ is positive in the interior of B^+_δ.

Proof of Lemma 4.4 : Let us recall that y is defined for $x \neq 0$ as

$$y_i = \frac{x_i}{x_N + r}, \text{ for } i = 1, \ldots, N-1.$$

This gives in particular, for $x_N \geq 0$

$$\frac{x_N}{r} = \frac{1-|y|^2}{1+|y|^2} \quad \text{and} \quad \frac{x_i}{r} = \frac{2}{1+|y|^2} y_i.$$

As we already pointed out before the image of $\{x_1 > 0, x_N > 0\} \cap \{|x| < R\}$ by this map is exactly the half ball in $\mathbb{R}^{N-1}$ defined by $B^+ = \{y_1 > 0\} \cap \{|y| < 1\}$. To be more correct one must consider, for δ small, the cone $S^+_{\delta,R} = \{\delta r \leq x_N\} \cap \{|x| < R\}$. Then for $\delta > 0$ given, using the interior sphere condition, one can choose R_δ small enough in order that $S^+_{\delta,R}$ be included in Ω. The image of this set by the map y is then the ball of $\mathbb{R}^{N-1}$ defined as $B^+_\delta = \{|y|^2 \leq \frac{1-\delta}{1+\delta}\}$. It is clear that $B^+_\delta \cap \{y_1 > 0\}$ is included in the half ball B^+ and tends to it.

We are going to introduce a second order operator on B^+ which is regular, fully nonlinear elliptic and we shall estimate its first eigenvalue on B^+. Then we shall use the properties of continuity of the eigenvalues and eigenfunctions with respect to the domain and the coefficients of the operator to obtain the result we want, that is to say that the first eigenvalue on B^+_δ for this operator is close to $2aN$ as soon as δ is small enough and a is close to A.

For simplicity we shall make the computation as if "$\delta = 0$" the changes to bring if one considers the set $S^+_{\delta,R}$ in place of $S^+ = S^+_{0,R}$ being obvious.

It is easy to see that for $i = 1, 2, \cdots N-1$,

$$\begin{aligned} r\nabla y_i &= \frac{e_i}{\frac{x_N}{r} + 1} - \frac{e_N + \frac{x}{r}}{(\frac{x_N}{r} + 1)^2}\frac{x_i}{r} \\ &= e_i(\frac{1+|y|^2}{2}) - (e_N + y)y_i. \end{aligned}$$

Hence,

$$r^2 \nabla y_i \cdot \nabla y_j = \delta_{ij}(\frac{1+|y|^2}{2})^2. \tag{4.6}$$

Furthermore

$$\begin{aligned} r^2 D^2 y_i &= \frac{-1}{(\frac{x_N}{r}+1)}[r\nabla y_i \otimes (e_N + \frac{x}{r}) + (e_N + \frac{x}{r}) \otimes r\nabla y_i + y_i \left(Id - \frac{x \otimes x}{r^2}\right)] \\ &= -\frac{1+|y|^2}{2}[e_i \otimes (e_N + y) + (e_N + y) \otimes e_i] + \frac{4y_i}{1+|y|^2}(e_N + y) \otimes (e_N + y) \\ &\quad + \frac{4y_i}{(1+|y|^2)^2}(2y + (1-|y|^2)e_N) \otimes (2y + (1-|y|^2)e_N) - \frac{y_i(1+|y|^2)Id}{2} \\ &:= M_i(y). \end{aligned}$$

Obvious direct calculations give, for $w(x) = r^\gamma \psi(y)$,

$$\nabla w = \gamma r^{\gamma-2}\psi x + r^\gamma \psi_{y_i} \nabla y_i,$$

hence $|\nabla w|\leq \gamma r^{\gamma-1}\psi + r^{\gamma-1}|\nabla\psi|$. Finally, summing over repeated indices

$$\begin{aligned} D^2 w &= r^{\gamma-2}\left(\psi_{y_i,y_j}(r\nabla y_i\otimes r\nabla y_j) + \right.\\ &\quad +\gamma\psi_{y_i}(\frac{x}{r}\otimes r\nabla y_i + r\nabla y_i\otimes\frac{x}{r})\\ &\quad +\psi_{y_i} r^2(D^2 y_i) + \\ &\quad \left. +\gamma\psi\left(I+(\gamma-2)\frac{x}{r}\otimes\frac{x}{r}\right)\right). \end{aligned}$$

Using the properties of the operator $\mathcal{M}^-_{a,A}$ and using the same notation for the Pucci's operators on matrices $N\times N$ and $(N-1)\times(N-1)$

$$\begin{aligned} \mathcal{M}^-_{a,A}(D^2 w) &\geq r^{\gamma-2}\left[\mathcal{M}^-_{a,A}(\psi_{y_iy_j}(r\nabla y_i\otimes r\nabla y_j)) + \right.\\ &\quad +\mathcal{M}^-_{a,A}(\psi_{y_i}(\gamma(\frac{x}{r}\otimes r\nabla y_i + r\nabla y_i\otimes\frac{x}{r}) + r^2 D^2 y_i)) + \\ &\quad \left. + a\gamma(N+\gamma-2)\psi\right], \end{aligned}$$

It is easy to see that the eigenvalues of the matrix $\psi_{y_iy_j} r^2(\nabla y_i\otimes\nabla y_j)$ are just the eigenvalues of $r^2 BD^2\psi$ where, $B=(b_{ij}),\ b_{ij}=\nabla y_i\cdot\nabla y_j$. Using (4.6),

$$r^2 B = \frac{I}{(\frac{x_N}{r}+1)^2} = \frac{I}{4}(1+|y|^2)^2$$

here I is the $(N-1)\times(N-1)$ identity matrix.

Finally, using $\nabla y_i\cdot x = 0$, one find that $(\frac{x}{r}\otimes r\nabla y_i + r\nabla y_i\otimes\frac{x}{r})$ has a null eigenvalue with multiplicity $N-2$ and two non zero eigenvalues $\lambda = \pm\frac{1}{\frac{x_N}{r}+1} = \frac{\pm 1}{2}(1+|y|^2)$.

These computations give

$$\begin{aligned} \mathcal{M}^-_{a,A}(D^2 w) &\geq r^{\gamma-2}\left[\frac{(1+|y|^2)^2}{4}\mathcal{M}^-_{a,A}(D^2\psi) + (a-A)\frac{\gamma}{2}(1+|y|^2)^2|\nabla\psi|\right.\\ &+ \left.\mathcal{M}^-_{a,A}(\psi_{y_i} r^2 D^2 y_i) + a\gamma(N+\gamma-2)\psi\right], \end{aligned} \tag{4.7}$$

where all the coefficients are continuous and bounded in $x_N\geq 0$. Remark that when $a=A$, the previous inequality is an equality, this is a key point for the arguments used later.

We now define, $H^\gamma_{a,A,\epsilon}$ to be the operator defined on $x_N>0$ by

$$\begin{aligned} H^\gamma_{a,A,\epsilon}(\psi) &:= \frac{(1+|y|^2)^2}{4}\mathcal{M}^-_{a,A}(D^2\psi)+\mathcal{M}^-_{a,A}(\psi_{y_i}M_i(y)) \\ &+ ((a-A)\frac{\gamma}{2}(1+|y|^2)^2-\epsilon)|\nabla\psi|. \end{aligned}$$

So that $H^\gamma_{a,A,\epsilon}$ is a fully nonlinear uniformly elliptic operator on the half ball $B^+(0,1)=\{y\in \mathbb{R}^{N-1}, |y|<1, y_1>0\}$ and it satisfies the hypothesis of the operators considered in [19] (see also [4], [22]). Let us recall the definition of the principal eigenvalue for $H^\gamma_{a,A,\epsilon}$ on the set $B^+(0,1)$

$$\overline{\lambda}(H^\gamma_{a,A,\epsilon})=\sup\{\lambda,\ \exists\ \psi>0 \text{ in } B^+(0,1),\ H^\gamma_{a,A,\epsilon}(\psi)+\lambda\psi\le\ 0 \text{ in } B^+(0,1)\}.$$

Using the results in [19], $\overline{\lambda}$ is well defined and there exists $\overline{\psi}>0$ in B^+ such that

$$H^\gamma_{a,A,\epsilon}(\overline{\psi})+\overline{\lambda}(H^\gamma_{a,A,\epsilon})\overline{\psi}=0$$

and $\overline{\psi}=0$ on ∂B^+. Furthermore $\overline{\psi}$ is Lipschitz continuous.

Let $g(\gamma,t,\epsilon)=\overline{\lambda}(H^\gamma_{a,ta,\epsilon})$ and $P(\gamma)=a\gamma(\gamma+N-2)$. Recall that by standard elliptic estimates, g is continuous. We want to prove that for any $\gamma>2$ (close to 2) there exist t_1 such that for $t\in(1,t_1)$ $g(\gamma,t,\epsilon)\le P(\gamma)$.

We first remark that for

$$t>1,\ \epsilon \text{ small enough } g(2,t,\epsilon)>2aN=P(2).$$

En fait c'est vrai quelque soit $\epsilon>0$ Indeed, let $w_1(x)=x_1x_N:=r^2\psi(y)$, by (4.7),

$$0>a-ta=\mathcal{M}^-_{a,ta}(D^2w)\ge H^2_{a,ta,0}(\psi)+2aN\psi.$$

Since $\psi>0$ in $S^+=\{y_1>0,|y|<1\}$, by definition of $\overline{\lambda}(H^2_{a,ta,0})$ one gets the result for $\epsilon=0$, we conclude using the increasing behaviour of g with respect to ϵ.

Furthermore $g(\gamma,1,0)=2aN$ for any γ. For $\gamma=2$ it is a consequence of the above calculation since ψ is the eigenvalue, but $H^\gamma_{a,a,0}$ is independent of γ hence it is true for any γ.

Take any $\gamma>2$,

$$\lim_{t\to1,\epsilon\to0} g(\gamma,t,\epsilon)+2\epsilon=2aN<P(\gamma)$$

then there exists $t_1 > 1$ and $\epsilon_o > 0$ such that

$$g(\gamma, t, \epsilon) + 2\epsilon < P(\gamma) \text{ for any } t \in (1, t_1) \text{ and any } \epsilon \in (0, \epsilon_o).$$

We now define $w = r^\gamma \psi$ where ψ is the principal positive eigenfunction for the operator $H^\gamma_{a,ta,\epsilon}$ in $B^+(0,1)$ with t and ϵ as above:

$$\begin{aligned}
\mathcal{M}^-_{a,A}(D^2 w) &\geq r^{\gamma-2}(H^\gamma_{a,ta,\epsilon}(\psi) + \epsilon|\nabla\psi| + a\gamma(N+\gamma-2)\psi) \\
&\geq r^{\gamma-2}[-\overline{\lambda}(H^\gamma_{a,ta,\epsilon})\psi + \ a\gamma(N+\gamma-2)\psi] + \epsilon r^{-1}|\nabla w| - \epsilon r^{-2} w \\
&= r^{-2}(-g(\gamma,t,\epsilon) + P(\gamma))w + \epsilon r^{-1}|\nabla w| - \epsilon r^{-2} w \\
&\geq \epsilon(r^{-2} w + r^{-1}|\nabla w|).
\end{aligned}$$

je ne comprends pas d'où vient le $\epsilon r^{-2} w$ deux lignes audessus, je pense qu'il n'y en a pas et alors on peut remplacer 2ϵ par ϵ dans l'inegalite 10 lignes au dessus This ends the proof of Lemma 4.4.

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