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Existence of solutions for semi-linear equations involving the p-Laplacian: the non coercive case

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1. Introduction

In this paper we give necessary and sufficient conditions for the existence of solutions of the following equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (g-\lambda)u^{p-1} = fu^{q-1},\ u \geq 0 \text{ in } \Omega \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N$, $p < q \leq \frac{pN}{N-p} := p^\star$, f and g belong to L^∞, and $\lambda \in \mathbb{R}$. By solution of (1.1), we mean a function $u \in W_0^{1,p}(\Omega)$ satisfying (1.1) in the weak usual sense.

In particular we shall study (1.1) considering the position of λ with respect to the principal eigenvalue. Precisely, it is well known that the concept of "eigenvalue" and "eigenfunction" has been generalized by many authors to the quasi-linear setting of the p-Laplacian $\Delta_p := \mathrm{div}(|\nabla.|^{p-2}\nabla.)$, in particular let us recall the works of Allegretto and Huang in [2], Anane in [3] and Lindqvist in [19]. We shall now state their definitions and the principal properties obtained in the works cited above.

Definition 1.1 *λ_1 the first "eigenvalue" of $-\mathrm{div}(|\nabla.|^{p-2}\nabla.) + g$ in $W_0^{1,p}(\Omega)$ is defined by*

$$\lambda_1 := \inf_{\{\psi \in W_0^{1,p}(\Omega), |\psi|_p = 1\}} \left\{ \int_\Omega |\nabla\psi|^p + \int_\Omega g|\psi|^p \right\}.$$

It is by now a classical result that there exists ϕ, positive in Ω for which this infimum is achieved. ϕ is called the "eigenfunction" corresponding to λ_1.

In particular ϕ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla\phi|^{p-2}\nabla\phi) + (g-\lambda_1)\phi^{p-1} = 0 \quad \text{in } \Omega \\ \phi = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Furthermore ϕ is simple, i.e. any solution of (1.2) satisfies $v = k\phi$ for some $k \in \mathbb{R}$. In the sequel we will normalize ϕ in the $L^p(\Omega)$ norm.

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Clearly for any $\lambda < \lambda_1$ the only nonnegative solution of

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (g-\lambda)u^{p-1} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is $u \equiv 0$.

On the other hand λ_1 is isolated, i.e. there exists $\delta > 0$ such that for any λ in $(\lambda_1, \lambda_1 + \delta)$ the only solution of (1.3) is $u \equiv 0$ as well.

Our first results concern some necessary conditions for the existence of solutions.

Theorem 1.2 *Suppose that there exists a nonnegative solution $u \not\equiv 0$ of equation (1.1). Then*

1) For $\lambda < \lambda_1$, the set Ω^+ defined as

$$\Omega^+ := \{x \in \Omega, f(x) > 0\}$$

is nonempty.

2) For $\lambda > \lambda_1$, $\Omega^- := \{x \in \Omega, f(x) < 0\} \neq \emptyset$ and $\int_\Omega f\phi^q < 0$.

3) For $\lambda = \lambda_1$, $\Omega^+ \neq \emptyset$, $\Omega^- \neq \emptyset$ and $\int_\Omega f\phi^q < 0$.

Theorem 1.3 *There exists $\lambda' > \lambda_1$ such that there are no non trivial non negative solutions of equation (1.1) for $\lambda > \lambda'$.*

Theorem 1.4 *Suppose that there exists $\bar{\lambda} > \lambda_1$ for which (1.1) possesses a solution. Then, (1.1) has a solution for $\lambda \in]\lambda_1, \bar{\lambda}]$.*

Our next result concerns the existence of solutions of equation (1.1) in the subcritical case:

Theorem 1.5 *Suppose that Ω^+ and Ω^- are nonempty, that $p < q < p^\star$, and $\int_\Omega f\phi^q < 0$. Then there exists $\delta > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ equation (1.1) has at least two non zero and nonnegative solutions of equation (1.1). For $\lambda = \lambda_1$ there exists at least one solution of (1.1) nonnegative and not identically zero.*

Remark 1. The solutions are obtained as minima of the two variational problems:

$$\alpha_{\lambda,q} = \inf_{\{u \in W_o^{1,p}(\Omega),\ \int_\Omega f|u|^q = -1\}} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)|u|^p \right\}$$

and

$$\mu_{\lambda,q} = \inf_{\{u \in W_o^{1,p}(\Omega),\ \int_\Omega f|u|^q = 1\}} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)|u|^p \right\}.$$

Indeed, if $u \in W_o^{1,p}(\Omega)$ realizes $\alpha_{\lambda,q}$ (respectively $\mu_{\lambda,q}$), so does $|u|$, and it is easy to see that u satisfies:

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (g-\lambda)u^{p-1} = -\alpha_{\lambda,q} f u^{q-1}$$

(respectively

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (g-\lambda)u^{p-1} = \mu_{\lambda,q} f u^{q-1}).$$

By a standard scaling argument one obtains two nonnegative solutions of equation (1.1), one being such that $\int_\Omega f u^q > 0$ and the other such that $\int_\Omega f u^q < 0$.

For simplicity of notation let $\alpha_\lambda := \alpha_{\lambda,p^\star}$ and $\mu_\lambda := \mu_{\lambda,p^\star}$.

Theorem 1.6 *Suppose that $q = p^\star$ and that $\Omega^+, \Omega^- \neq \emptyset$, that $\lambda > \lambda_1$ and that $\int_\Omega f\phi^{p^\star} < 0$. Then there exists $\delta > 0$ such that if $\lambda \in (\lambda_1, \lambda_1 + \delta)$ there exists at least one solution of equation (1.1). If moreover,*

$$\mu_\lambda < K(N,p)^{-p} \sup |f|^{\frac{-p}{p^\star}},$$

then, there exist at least two non zero solutions of equation (1.1).

Remark 2. As in the subcritical case, the solutions are obtained as minima of α_λ and μ_λ.

Remark 3. According to Theorems 1.4 and 1.5 the solutions of equation (1.1) exist for an interval, $(\lambda_1, \bar{\lambda})$. On the other hand for some $\lambda \in]\lambda_1, \bar{\lambda}[$, there may be only one solution, because for λ not close to λ_1 nothing can be said about the sign of $\int_\Omega f u_\lambda^q$ when u_λ is a solution obtained by Theorem 1.4.

For $p = 2$ i.e. the classical Laplacian and $2 < q < \dfrac{2n}{n-2}$ problem (1.1) has been extensively studied when $f > 0$. Since we are concerned with the case where f changes sign, let us recall the main results in that case. Necessary and sufficient conditions for the existence of solutions for (1.1) have been given by Alama and Tarantello [1], Berestycki, Capuzzo Dolcetta and Nirenberg [5] and Ouyang [20] in the non coercive case.

Alama and Tarantello in [1] and the authors of the present paper in [6] have studied the critical case i.e. $q = \frac{2n}{n-2}$. Let us also mention the very interesting work of Chen and Li in [7].

It is well known that the p-Laplacian appears in many contexts : Non-Newtonian fluids, nonlinear elasticity and reaction diffusion problems just to name a few. Indeed equation (1.1) has been extensively studied for general p and q; in particular for q critical, existence of solutions of problem (1.1) was studied by Guedda and Veron in [14] for $f \equiv 1, g(x) \equiv \lambda = 0$. Demengel and Hebey in [10] gave existence of variational solutions when f changes sign and the functional $\int_\Omega |\nabla u|^p + \int_\Omega (g - \lambda)|u|^p$ is coercive i.e. $\lambda < \lambda_1$.

In [12], the authors study a similar problem with $(g - \lambda)u^{p-1}$ replaced by $cteu^{k-1}$ with $k \neq p$.

Always for general p but q subcritical the non coercive case was also studied by Drabek and Pohozaev in [11]; they use the fibering method to obtain some existence results for λ close to λ_1. See also Pohozaev and Veron [21] for the Neumann problem.

Finally for q critical, Drabek and Huang studied the problem in $\mathbb{R}^N$ [10], while Arioli and Gazzola in [4] proved existence for solutions changing sign through a linking method.

The above Theorems are the natural extension to the p-Laplacian of the results obtained in [6]. Nonetheless the proofs differ from the case $p = 2$. In particular the proofs of Theorems 1.5 and 1.6 follow the approach taken by Ouyang in [20]. Although we should mention that Ouyang treats the sub-critical case and he uses bifurcation technic that don't hold for $p \neq 2$.

The outline of the paper is the following. In the next section we prove the necessary conditions (i.e. Theorem 1.2 and 1.3) using among other things Picone's identity for the p-Laplacian (cf Allegretto and Huang [2]). In the third section we prove the existence results first for the sub-critical case and then for the critical case. Finally in the last section we construct some test functions to show that the condition on μ_λ of Theorem 1.6 can be satisfied and easily verified.

2. Proofs of Theorem 1.2, 1.3, 1.4.

Let us recall Picone's identity for the p-Laplacian as formulated by Allegretto and Huang in [2]. Suppose that v and w belong to $W^{1,p}(\Omega)$ with $v \geq 0$ and $w > 0$, then

$$|\nabla v|^p - \nabla\left(\frac{v^p}{w^{p-1}}\right) \cdot \sigma(w) \geq 0$$

everywhere in Ω, for $\sigma(w) := |\nabla w|^{p-2}\nabla w$.

Moreover if equality holds then $w = kv$ for some constant $k \in \mathbb{R}$.

Proof of Theorem 1.2. Since in the case $\lambda < \lambda_1$ the functional

$$I_\lambda(u) := \int_\Omega |\nabla u|^p + \int_\Omega (g - \lambda)|u|^p$$

is coercive the first assertion is obvious.

Let us prove 2. Suppose that $\lambda > \lambda_1$, and let u be a nonnegative solution of (1.1) . Adapting the strict maximum principle of Vasquez, one has $u > 0$ inside Ω . In addition, from regularity results of [13], [23], [17], [9], u is $\mathcal{C}^{1,\alpha}(\bar{\Omega})$, for every $\alpha \in [0, 1[$. Using once more the strict maximum principle inspired from Hopf's lemma, as given in [24], one has the existence of some real $\epsilon > 0$ such that $\phi \geq \epsilon u$ on $\bar{\Omega}$. As a consequence, one is allowed to multiply the equation (1.1) by $(u)^{1-q}\phi^q$. Integrating by parts on Ω, one obtains

$$\begin{aligned}\int_\Omega f\phi^q &= \int_\Omega \sigma(u).\nabla(u^{1-q}\phi^q) + \int_\Omega (g-\lambda)u^{p-1}u^{1-q}\phi^q \\ &= (1-q)\int_\Omega |\nabla u|^p \left(\frac{\phi}{u}\right)^q + q\int_\Omega (\sigma(u).\nabla\phi)\left(\frac{\phi}{u}\right)^{q-1} \\ &+ \int_\Omega (g-\lambda)u^{p-q}\phi^q. \end{aligned} \tag{2.4}$$

Now we multiply equation (1.2) by $\phi^{q-p+1}u^{p-q}$ and integrate over Ω;

$$\int_\Omega \sigma(\phi).\nabla(\phi^{q-p+1}u^{p-q}) + \int_\Omega (g-\lambda_1)\phi^q u^{p-q} = 0$$

and then

$$\begin{aligned}(q-p+1)\int_\Omega |\nabla\phi|^p \left(\frac{\phi}{u}\right)^{q-p} + (p-q)\int_\Omega \sigma(\phi).\nabla u\left(\frac{\phi}{u}\right)^{q-p+1} + \\ + \int_\Omega (g-\lambda_1)\phi^q u^{p-q} = 0. \end{aligned} \tag{2.5}$$

Subtracting (2.4) to (2.5) , one gets

$$(q-p+1)\int_\Omega |\nabla\phi|^p\left(\frac{\phi}{u}\right)^{q-p} + (p-q)\int_\Omega \sigma(\phi).\nabla u\left(\frac{\phi}{u}\right)^{q-p+1}$$
$$-q\int_\Omega \left(\frac{\phi}{u}\right)^{q-1}\nabla\phi.\sigma(u) + (q-1)\int_\Omega \left(\frac{\phi}{u}\right)^{q}|\nabla u|^p +$$
$$(\lambda-\lambda_1)\int_\Omega \phi^q u^{p-q} = -\int_\Omega f\phi^q. \tag{2.6}$$

Now apply Picone's identity as follows

$$|\nabla u|^p - \nabla\left(\frac{u^p}{\phi^{p-1}}\right)\cdot\sigma(\phi)\geq 0.$$

Multiplying it by $\left(\frac{\phi}{u}\right)^q$ and integrating over Ω it becomes

$$\int_\Omega |\nabla u|^p\left(\frac{\phi}{u}\right)^q - p\int_\Omega \nabla u\cdot\sigma(\phi)u^{p-q-1}\phi^{q-p+1} +$$
$$+ (p-1)\int_\Omega |\nabla\phi|^p u^{p-q}\phi^{q-p}\geq 0. \tag{2.7}$$

Similarly, exchanging the role of u and ϕ i.e. considering

$$|\nabla\phi|^p - \nabla(\frac{\phi^p}{u^{p-1}})\cdot\sigma(u)\geq 0$$

and multiplying by $\left(\frac{\phi}{u}\right)^{q-p}$ one gets

$$\int_\Omega |\nabla\phi|^p\left(\frac{\phi}{u}\right)^{q-p} - p\int_\Omega \left(\frac{\phi}{u}\right)^{q-1}\nabla\phi.\sigma(u)$$
$$+ (p-1)\int_\Omega \left(\frac{\phi}{u}\right)^{q}|\nabla u|^p\geq 0. \tag{2.8}$$

Multiply (2.8) by $\frac{q}{p}$ and (2.7) by $\frac{q}{p}-1$ their sum gives

$$(q-p+1)\int_\Omega |\nabla\phi|^p\left(\frac{\phi}{u}\right)^{q-p} + (p-q)\int_\Omega \nabla u\cdot\sigma(\phi)\left(\frac{\phi}{u}\right)^{q-p+1} +$$
$$-q\int_\Omega \left(\frac{\phi}{u}\right)^{q-1}\nabla\phi\cdot\sigma(u) + (q-1)\int_\Omega |\nabla u|^p\left(\frac{\phi}{u}\right)^{q}\geq 0. \tag{2.9}$$

Substracting (2.9) from (2.6) we obtain

$$\int_\Omega f\phi^q + (\lambda-\lambda_1)\int_\Omega \phi^q u^{p-q}\leq 0. \tag{2.10}$$

When $\lambda > \lambda_1$, this implies that $\int_\Omega f\phi^q < 0$ and 2) is proved.

For the proof of 3), let $\lambda = \lambda_1$ and let u be a nonnegative solution of equation (1.1). Multiplying it by u one obtains

$$\int_\Omega |\nabla u|^p + \int_\Omega (g - \lambda_1)u^p = \int_\Omega fu^q.$$

Since the functional I_{λ_1} is non negative, one has $\int_\Omega fu^q \geq 0$. Suppose that it is zero. Then u would be an eigenfunction for the eigenvalue λ_1, which would imply that $fu^{q-1} = 0$. Then u must be zero on a set of positive measure, which contradicts the fact that u is parallel to $\phi > 0$ in Ω. We have proved that $\int_\Omega fu^q > 0$, this implies that $\Omega^+ \neq \emptyset$.

We shall now prove that $\int_\Omega f\phi^q < 0$, this of course implies also that $\Omega^- \neq \emptyset$.

From the previous computations in the proof of 2), and precisely from (2.6) with $\lambda = \lambda_1$ and from (2.9), we obtain that

$$\begin{aligned}(q-p+1)\int_\Omega |\nabla\phi|^p \left(\frac{\phi}{u}\right)^{q-p} + (p-q)\int_\Omega \nabla u\cdot\sigma(\phi)\left(\frac{\phi}{u}\right)^{q-p+1}\\ -q\int_\Omega \left(\frac{\phi}{u}\right)^{q-1}\nabla\phi\cdot\sigma(u) + (q-1)\int_\Omega |\nabla u|^p\left(\frac{\phi}{u}\right)^q +\\ = -\int_\Omega f\phi^q. \qquad (2.11)\end{aligned}$$

As a consequence $\int_\Omega f\phi^q \leq 0$. Suppose by contradiction that $\int_\Omega f\phi^q = 0$, then the left hand side of the previous identity is zero. Recalling (2.8) and (2.9) the left hand side is a sum of two nonnegative quantities, hence they must be both null. Therefore we have obtained that

$$\begin{aligned}\int_\Omega |\nabla\phi|^p\left(\frac{\phi}{u}\right)^{q-p} - p\int_\Omega \left(\frac{\phi}{u}\right)^{q-1}\nabla\phi.\sigma(u)\\ + (p-1)\int_\Omega \left(\frac{\phi}{u}\right)^q |\nabla u|^p = 0 \qquad (2.12)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\nabla u|^p\left(\frac{\phi}{u}\right)^q - p\int_\Omega \nabla u\cdot\sigma(\phi)u^{p-q-1}\phi^{q-p+1} +\\ + (p-1)\int_\Omega |\nabla\phi|^p u^{p-q}\phi^{q-p} = 0. \qquad (2.13)\end{aligned}$$

Clearly (2.12) and (2.13) imply that

$$|\nabla u|^p - \nabla\left(\frac{u^p}{\phi^{p-1}}\right)\cdot\sigma(\phi) = 0$$

and

$$|\nabla\phi|^p - \nabla\left(\frac{\phi^p}{u^{p-1}}\right)\cdot\sigma(u) = 0.$$

Each of these identities implies that ϕ is parallel to u. Then u is an eigenfunction. This implies that fu^{q-1} is identically zero which is a contradiction. □

Proof of Theorem 1.3. Let B be a ball on which $f > 0$, $B \subset\subset \Omega^+$. Let then $(\psi, \mu^\star)$ be the non zero and non negative normalized solution, of

$$\begin{cases} -\Delta_p \psi + (-\mu^\star)\psi^{p-1} = 0 & \text{in } B \\ \psi = 0 & \text{on } \partial B. \end{cases}$$

Suppose that a solution of equation (1.1) exists for λ such that $|g|_\infty + \mu^\star < \lambda$, $u \geq 0$ and non identically zero. On B, by the strict maximum principle of Vasquez, $u > 0$. Using Picone's identity, one has

$$|\nabla \psi|^p - \nabla \left(\frac{\psi^p}{u^{p-1}} \right) .\sigma(u) \geq 0$$

in B, hence, integrating over B

$$0 \leq \int_B (\mu^\star)\psi^p + \int_B (g - \lambda)\psi^p \tag{2.14}$$

here, we have used the fact that $\psi = 0$ on ∂B and the equation verified by u, since

$$-\Delta_p u + (g - \lambda)u^{p-1} = fu^{q-1} \geq 0$$

on B. (2.14) of course contradicts the choice of λ. □

Proof of Theorem 1.4. Let $\bar\lambda$ be such that $\lambda_1 < \bar\lambda$ and take $\lambda \in]\lambda_1, \bar\lambda[$. Let $\bar u$ be a solution of (1.1) for $\bar\lambda$. Then $\bar u$ is a supersolution of (1.1) for λ. Indeed

$$-\Delta_p \bar u + (g - \lambda)\bar u^{p-1} = f\bar u^{q-1} + (\bar\lambda - \lambda)\bar u^{p-1} \geq f\bar u^{q-1}$$

and $\bar u = 0$ on the boundary. On another hand, taking ϵ small enough, $\epsilon\phi$ is a subsolution, since

$$-\Delta_p(\epsilon\phi) + (g - \lambda)(\epsilon\phi)^{p-1} = (\lambda_1 - \lambda)\epsilon^{p-1}\phi^{p-1} \leq f\epsilon^{q-1}\phi^{q-1},$$

(using $p < q$ and $(\lambda_1 - \lambda)\epsilon^{p-1}\phi^{p-1} < 0$). Moreover, using strong maximum principle of Vasquez and regularity results, one can choose ϵ small enough in order to have $\bar u \geq \epsilon\phi$. Finally we use the following Proposition, whose proof can be found in the appendix and is a mere adaptation of the classical sub and super solution for $p = 2$. (see e.g. [15], see also [22]):

Proposition 2.1 *Suppose that $f(x,t) = a(x)|t|^{q-2}t + b(x)|t|^{p-2}t$ with $1 < p < q$ with a and b two continuous and bounded functions on Ω Suppose that $\bar u$ is a weak supersolution for $-\Delta_p u + f(x,u)$, $\bar u = 0$ on $\partial\Omega$, and that $\underline{u}$ is a weak subsolution with $\underline{u} = 0$ on $\partial\Omega$. Suppose that there exists some constant c and C such that*

$$-\infty < c \leq \underline{u} \leq \bar u \leq C < +\infty$$

Then, there exists a solution u between $\underline{u}$ and $\bar u$

Using this Proposition with $f(x,u) = (g - \lambda)u^{p-1} - fu^{q-1}$, and $\underline{u} = \epsilon\phi$, one obtains that there exists a solution which is such that

$$\epsilon\phi \leq u \leq \bar u.$$

3. Existence of solutions

Proof of Theorem 1.5. This proof is inspired by the arguments used in [20]. We begin with the subcritical case. Suppose that $q < p^\star$. Let us recall the following notations:

$$\lambda_q^\star = \inf_{\{u\in W_0^{1,p}(\Omega),\ |u|_p^p=1, \int_\Omega fu^q=0\}} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)|u|^p \right\}$$

$$\alpha_{\lambda,q} = \inf_{\{u,\ \int_\Omega f|u|^q=-1\}} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)|u|^p \right\} \tag{3.15}$$

and

$$\mu_{\lambda,q} = \inf_{\{u,\ \int_\Omega f|u|^q=1\}} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)|u|^p \right\}. \tag{3.16}$$

Let $I_\lambda(u) := \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)|u|^p$.

We will prove the following facts

1. $\lambda_q^\star > 0$.
2. For $\lambda \in]\lambda_1, \lambda_1+\lambda_q^\star[$, $\alpha_{\lambda,q} < 0$ and it is achieved; $\alpha_{\lambda_1,q} = 0$.
3. For $\lambda \in]\lambda_1, \lambda_1+\lambda_q^\star[$, $\mu_{\lambda,q} > 0$ and it is achieved. Moreover $\mu_{\lambda_1,q} > 0$.

Proof of 1. By the definition of λ_1, $\lambda_q^\star \geq 0$. Suppose by contradiction that $\lambda_q^\star = 0$. Let (u_n) be a minimizing sequence. Since $|\nabla |u_n|| = |\nabla u_n|$, one can assume that $u_n \geq 0$. Since $|u_n|_p = 1$ and $\int_\Omega |\nabla u_n|^p + \int_\Omega (g-\lambda_1)u_n^p \to 0$, then $\int_\Omega |\nabla u_n|^p$ is bounded; hence (u_n) is bounded in $W_0^{1,p}$. Extracting from it a subsequence and passing to the limit, one gets that there exists some $u \geq 0$, weak limit of (u_n) in $W^{1,p}(\Omega)$, such that

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)u^p \leq 0. \tag{3.17}$$

Clearly (3.17) implies that

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)u^p = 0.$$

and then u is an eigenfunction for λ_1 and then it is parallel to ϕ. Moreover $u \in W_0^{1,p}$, $\int_\Omega |u|^p = 1$ and $\int_\Omega fu^q = 0$, which contradicts the assumption $\int_\Omega f\phi^q < 0$. Finally $\lambda_q^\star > 0$.

Proof of 2. In order to prove that $\alpha_{\lambda,q} < 0$ for $\lambda > \lambda_1$, let us take, as an admissible function, $v = \dfrac{\phi}{(-\int_\Omega f\phi^q)^{\frac{1}{q}}}$. We then have

$$\alpha_{\lambda,q} \leq I_\lambda(v) = \frac{1}{(-\int_\Omega f\phi^q)^{\frac{p}{q}}} I_\lambda(\phi) = \frac{1}{(-\int_\Omega f\phi^q)^{\frac{p}{q}}}(\lambda_1 - \lambda) < 0.$$

Now we will check that

$$\alpha_{\lambda,q} > -\infty.$$

If not, there would exist a subsequence (u_i), $u_i \geq 0$ for all i, such that $\int_\Omega f u_i^q = -1$ and $I_\lambda(u_i) \to -\infty$. Clearly $|u_i|_p \to +\infty$ since

$$\overline{\lim} \int_\Omega (g-\lambda) u_i^p \leq \alpha_{\lambda,q}.$$

Let $w_i = \dfrac{u_i}{|u_i|_p}$. One has $\int_\Omega f w_i^q \to 0$, and (w_i) is bounded in $W_0^{1,p}(\Omega)$, since $|w_i|_p = 1$ and $\displaystyle\int_\Omega |\nabla w_i|^p + \int_\Omega (g-\lambda) w_i^p = \frac{I_\lambda(u_i)}{|u_i|_p^p} \leq 0$ implies

$$\int_\Omega |\nabla w_i|^p \leq |g-\lambda|_\infty.$$

Then, there exists a subsequence still denoted (w_i), such that $w_i \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$. Observe that

$$\int_\Omega |w|^p = 1 \text{ and } I_\lambda(w) \leq 0.$$

This contradicts the definition of λ, since $\int_\Omega f w^q = 0$ and $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^\star[$. We have proved that $\alpha_{\lambda,q} > -\infty$.

We shall now see that $\alpha_{\lambda,q}$ is achieved. Let (u_n), $u_n \geq 0$ be a minimizing sequence for $\alpha_{\lambda,q}$ i.e.

$$\int_\Omega |\nabla u_n|^p + \int_\Omega (g-\lambda) u_n^p \to \alpha_{\lambda,q},$$

$$\int_\Omega f u_n^q = -1.$$

Let us prove first that $|u_n|_p$ is bounded. If not, one can argue as previously by considering $w_n = \dfrac{u_n}{|u_n|_p}$. It is easy to see that (w_n) converges weakly in $W^{1,p}(\Omega)$, up to a subsequence, towards some function $w \geq 0$ which satisfies $\int_\Omega f w^q = 0$, $|w|_p = 1$ and

$$\int_\Omega |\nabla w|^p + \int_\Omega (g-\lambda) w^p = 0.$$

This contradicts the definition of λ. Hence $\int_\Omega |u_n|^p$ is bounded, and so is $\int_\Omega |\nabla u_n|^p$. By extracting from (u_n) a subsequence, one obtains that there exists $u \in W_0^{1,p}$, $u \geq 0$, such that $\int_\Omega f u^q = -1$ and by lower semi-continuity of the semi-norm $|\nabla u|_p$ with respect to the weak topology,

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda) u^p \leq \alpha_{\lambda,q}.$$

Finally using the definition of $\alpha_{\lambda,q}$, u is a minimizer for $\alpha_{\lambda,q}$, hence it is a nonzero solution of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + (g-\lambda)u^{p-1} = -\alpha_{\lambda,q} f u^{q-1}.$$

Proof of 3. Acting as we did for $\alpha_{\lambda,q}$ one can prove that $\mu_{\lambda,q} > -\infty$. We are now going to check that $\mu_{\lambda,q}$ is achieved.

Indeed, let u_n be a sequence such that $u_n \geq 0$,

$$\int_\Omega |\nabla u_n|^p + \int_\Omega (g-\lambda)u_n^p \to \mu_{\lambda,q},$$

$$\int_\Omega f u_n^q = 1.$$

Suppose that $|u_n|_p \to \infty$. Then considering $w_n = \dfrac{u_n}{|u_n|_p}$ one gets, by passing to the limit that there exists $w \geq 0$, a weak limit of (w_n) in $W^{1,p}(\Omega)$, such that

$$\int_\Omega |\nabla w|^p + \int_\Omega (g-\lambda)w^p \leq 0$$

and $\int_\Omega f w^q = 0$, which contradicts the assumption $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^\star[$. Then (u_n) is bounded and we pass to the limit to obtain

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)u^p = \mu_{\lambda,q}$$

and $\int_\Omega f u^q = 1$. Hence $\mu_{\lambda,q}$ is achieved.

For $\lambda = \lambda_1$, $\mu_{\lambda_1,q} \geq 0$, but since it is achieved, if $\mu_{\lambda_1,q} = 0$, we would have an eigenfunction u such that $\int_\Omega f u^q = 1$, which contradicts the assumptions. Then $\mu_{\lambda_1,q} > 0$.

For $\lambda > \lambda_1$ let $u_q \geq 0$ which realizes the minimum in $\mu_{\lambda,q}$. Then :

$$-\Delta_p u_q + (g-\lambda)u_q^{p-1} = \mu_{\lambda,q} f u_q^{q-1}.$$

Using the procedure of the proof of Theorem 1.2 for u_q, inequality (2.10) becomes

$$\mu_{\lambda,q} \int_\Omega f\phi^q + (\lambda - \lambda_1) \int_\Omega \phi^q u_q^{p-q} \leq 0.$$

Using $\int_\Omega f\phi^q < 0$ and $\lambda - \lambda_1 > 0$, one gets $\mu_{\lambda,q} > 0$. □

Let us now state and prove some results concerning $\alpha_{\lambda,q}$ and $\mu_{\lambda,q}$.

Lemma 3.1 *The following convergences hold:*

$$\lim_{\lambda \to \lambda_1} \alpha_{\lambda,q} = \alpha_{\lambda_1,q} = 0, \tag{3.18}$$

$$\lim_{\lambda \to \lambda_1} \mu_{\lambda,q} = \mu_{\lambda_1,q} \tag{3.19}$$

Lemma 3.2 *1.* $\lambda^\star_{p^\star} \geq \overline{\lim}_{q\to p^\star}\lambda^\star_q \geq \underline{\lim}_{q\to p^\star}\lambda^\star_q := \lambda^\star > 0.$

2. For $\lambda_1 \leq \lambda < \lambda_1 + \lambda^\star$, *then* $0 \leq \underline{\lim}_{q\to p^\star}\mu_{\lambda,q} \leq \overline{\lim}_{q\to p^\star}\mu_{\lambda,q} \leq \mu_\lambda(= \mu_{\lambda,p^\star})$.

3. For λ *close to* λ_1, $\alpha_\lambda(=\alpha_{\lambda,p^\star}) > -\infty$ *and* $\overline{\lim}_{q\to p^\star}\alpha_{\lambda,q} \leq \alpha_\lambda$.

Proof of Lemma 3.1. Suppose by contradiction that (3.18) does not hold, then there exist some number $\alpha < 0$ and a sequence of $\lambda \in \mathbb{R}$, $\lambda \to \lambda_1$, and $(u_\lambda) \subset W^{1,p}_o(\Omega)$ such that

$$\int_\Omega |\nabla u_\lambda|^p + \int_\Omega (g-\lambda)|u_\lambda|^p \leq \alpha.$$

Moreover one can assume that $u_\lambda \geq 0$. If (u_λ) is bounded, we may extract from it a subsequence weakly convergent to some $u \in W^{1,p}_0$, such that

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)u^p \leq \alpha < 0,$$

which is absurd.

On the other hand if (u_λ) diverges we can normalize it and then we obtain a sequence (w_λ) such that $\int_\Omega |w_\lambda|^p = 1$. By extracting a subsequence, there exists $w \geq 0$, such that $\int_\Omega |w|^p = 1$, $\int_\Omega f w^q = 0$ and

$$\int_\Omega |\nabla w|^p + \int_\Omega (g-\lambda_1)w^p \leq 0.$$

This would imply that w is parallel to ϕ which is absurd since $\int_\Omega f\phi^q < 0$.

Let us now prove (3.19). Let us define $\bar\mu_q := \overline{\lim}_{\lambda\to\lambda_1}\mu_{\lambda,q}$. One already has $\bar\mu_q \leq \mu_{\lambda_1,q}$. Let u_λ which satisfies $u_\lambda \geq 0$ and

$$-\Delta_p u_\lambda + (g-\lambda)u_\lambda^{p-1} = \mu_{\lambda,q} f u_\lambda^{q-1} \tag{3.20}$$

$$\int_\Omega f u_\lambda^q = 1.$$

As we did above , one can prove that (u_λ) is bounded in the $W^{1,p}$ norm. By extracting a subsequence, one gets by passing to the limit when $\lambda \to \lambda_1$

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)u^p \leq \bar\mu_q$$

and $u \geq 0$, $\int_\Omega f u^q = 1$. This clearly implies that $\bar\mu_q \geq \mu_{\lambda_1,q}$ and gives the required result. □

Proof of Lemma 3.2. Let us prove 1, and first that $\underline{\lim}_{q\to p^\star}\lambda^\star_q > 0$. Since $\lambda^\star_q$ is achieved, let $u_q \geq 0$ be a solution of

$$\int_\Omega |\nabla u_q|^p + \int_\Omega (g-\lambda_1)u_q^p = \lambda^\star_q$$

$|u_q|_p = 1$ and $\int_\Omega f u^q = 0$. Suppose by contradiction that $\underline{\lim}_{q\to p^\star} \lambda_q^\star = 0$. Then, by extracting from (u_q) a subsequence, one gets by passing to the limit when q tends to $p^\star$:

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)u^p \le 0$$

and $|u|_p = 1$. Since I_{λ_1} is coercive, $\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda_1)u^p = 0$, and the sequence $\int_\Omega |\nabla u_q|^p$ tends to $\int_\Omega |\nabla u|^p$. Hence u_q tends to u strongly in $W^{1,p}(\Omega)$, and finally $\int_\Omega f u^{p^\star} = \lim_{q\to p^\star} \int_\Omega f u_q^q = 0$. This is a contradiction since ϕ is simple and $\int_\Omega f\phi^{p^\star} < 0$. As a consequence $\lambda^\star > 0$.

We now prove that $\lambda^\star \le \lambda_{p^\star}^\star$. Indeed, let $u \ge 0$ be a $\mathcal{C}^1$ function, such that $\int_\Omega f u^{p^\star} = 0$, $|u|_p = 1$, and

$$I_{\lambda_1}(u) \le \lambda_{p^\star}^\star + \epsilon.$$

If there exists an infinite sequence $q \to p^\star$, such that $\int_\Omega f u^q = 0$, one has the desired result. If not, there exists an infinite sequence $q \to p^\star$ such that either $\int_\Omega f u^q > 0$ for all q, or $\int_\Omega f u^q < 0$ for all q. Suppose that we are in the first case and define $\alpha(q) = \dfrac{\int_\Omega f u^q}{\int_\Omega f u^q - \int_\Omega f\phi^q}$. Then $\alpha(q) \in [0,1]$, and $\alpha(q) \to 0$ when $q \to p^\star$. Let us define

$$v_q = (\alpha(q)\phi^q + (1-\alpha(q))u^q)^{\frac{1}{q}}\ .$$

By the regularity properties of ϕ and u, v_q belongs to $W_0^{1,p}(\Omega)$, $v_q \ge 0$ and $\int_\Omega f v_q^q = 0$ by the choice of $\alpha(q)$. Moreover it is easy to check that v_q tends to u in $W^{1,p}(\Omega)$ strongly. As a consequence

$$\lambda_q^\star(1+o(1)) \le \lambda_q^\star \left(\int_\Omega v_\alpha^p\right) \le \int_\Omega |\nabla v_\alpha|^p + \int_\Omega (g-\lambda_1)v_\alpha^p \le \lambda_{p^\star}^\star + \epsilon + o(1)$$

when $q \to p^\star$.This implies that $\lambda^\star \le \lambda_{p^\star}^\star$. Suppose now that there exists a sequence $q \to p^\star$ such that $\int_\Omega f u^q < 0$. Let u_0 be nonnegative in $\mathcal{C}^1(\bar\Omega)$, such that $\int_\Omega f u_0^q > 0$ and define

$$v_\alpha = (\alpha(q)u_0^q + (1-\alpha(q))u^q)^{\frac{1}{q}}\ ,$$

where $\alpha(q) = \dfrac{\int_\Omega f u^q}{\int_\Omega f u^q - \int_\Omega f u_0^q}$. One concludes as in the case $\int_\Omega f u^q > 0$.

To prove 2., let $\varepsilon > 0$ be given and let u be such that $u \ge 0$, $\int_\Omega f u^{p^\star} = 1$ and

$$I_\lambda(u) \le \mu_\lambda + \varepsilon.$$

Then for q close to $p^\star$, $\int_\Omega f u^q > \frac{1}{2}$ and taking $v_q = \dfrac{u}{(\int_\Omega f u^q)^{\frac{1}{q}}}$, one gets, for q sufficiently close to p^*,

$$\mu_{\lambda,q} \le I_\lambda(v_q) \le \mu_\lambda + 2\varepsilon.$$

We will prove 3. by contradiction. Hence suppose that there exists a sequence $\lambda_n \to \lambda_1$ and a sequence (u_n), $u_n \geq 0$ such that $\int_\Omega f u_n^{p^\star} = -1$ and $I_{\lambda_n}(u_n) \leq -n$. Clearly $|u_n|_p \to +\infty$. Then defining $w_n = \dfrac{u_n}{|u_n|_p}$, and extracting a subsequence from it, one gets that there exists $w \geq 0$ such that

$$I_{\lambda_1}(w) \leq 0.$$

This in fact implies that strong convergence holds and then $\int_\Omega f w^{p^\star} = 0$, which contradicts $|w|_p = 1$ and ϕ is simple. □

Before giving the proof of Theorem 1.6 let us recall one of the key ingredients employed herein i.e. the famous concentration compactness principle of P. L. Lions [18]:

Lemma 3.3 *Let Ω be some bounded open set in $\mathbb{R}^n$, and (u_k) be some sequence in $W_o^{1,p}(\Omega)$, which is bounded in $W^{1,p}(\Omega)$. Then there exist a subsequence of (u_k), still denoted (u_k) for simplicity, two nonnegative measures μ and ν on $\overline{\Omega}$, a sequence of points x_i in $\overline{\Omega}$, two sequences of nonnegative real numbers μ_i and ν_i and a function u in $W_o^{1,p}(\Omega)$, such that*

$$|\nabla u_k|^p \rightharpoonup \mu \geq |\nabla u|^p + \sum_i \mu_i \delta_{x_i}$$

(the convergence being tight on $\overline{\Omega}$ i.e. $\int_\Omega |\nabla u_k|^p \to \int_{\overline{\Omega}} \mu$,),

$$|u_k|^{p^\star} \rightharpoonup \nu = |u|^{p^\star} + \sum_i \nu_i \delta_{x_i}$$

(the convergence being tight on $\overline{\Omega}$ i.e. $\int_\Omega |u_k|^{p^\star} \to \int_{\overline{\Omega}} \nu$), with the inequality

$$\nu_i \leq K(n,p)^{\frac{p^\star}{p}} \mu_i. \tag{3.21}$$

Proof of Theorem 1.6.
First part. We prove the existence of solutions for α_λ and for λ sufficiently close to λ_1. According to Lemma 3.1 above, $\lim\limits_{\lambda \to \lambda_1} \alpha_\lambda = 0$. One takes λ sufficiently close to λ_1 in order to have $-\alpha_\lambda < K(N,p)^{-p}(\sup|f|)^{\frac{-p}{p^\star}}$, and $\lambda < \lambda_1 + \lambda^\star$. Let (u_q), $u_q \geq 0$ be a solution for the problem defining $\alpha_{\lambda,q}$.

Claim. $(u_q)_q$ is bounded in L^p.

Suppose that it is not true. Then, proceeding as in the proof of Theorem 1.5, there would exist a sequence (w_q) such that $w_q \geq 0$, $|w_q|_p = 1$, and

$$\int_\Omega |\nabla w_q|^p + \int_\Omega (g - \lambda) w_q^p \leq 0. \tag{3.22}$$

Extracting from (w_q) a subsequence one obtains that there exists w, weak limit of w_q in $W^{1,p}$ such that $w \geq 0$, $|w|_p = 1$, and

$$\int_\Omega |\nabla w|^p + \int_\Omega (g-\lambda) w^p \leq 0.$$

If $\int_\Omega f w^{p^\star} = 0$, this contradicts the assumption $\lambda < \lambda_1 + \lambda^\star \leq \lambda_1 + \lambda^\star_{p^\star}$. If $\int_\Omega f w^{p^\star} > 0$, $I_\lambda(w) \geq \mu_\lambda (\int_\Omega f w^{p^\star})^{\frac{p}{p^\star}} > 0$, and since $\mu_\lambda \geq 0$ one would obtain that $\mu_\lambda = 0 = I_\lambda(w)$, and using lower semi-continuity for the weak topology

$$I_\lambda(w) \leq \underline{\lim}_{q\to p^\star} I_\lambda(w_q) \leq 0.$$

Finally $I_\lambda(w) = \lim_{q\to p^\star} I_\lambda(w_q)$ and then $\int_\Omega |\nabla w_q|^p \to \int_\Omega |\nabla w|^p$, strong convergence holds in fact, hence $\int_\Omega f w^{p^\star} = \lim_{q\to p^\star} \int_\Omega f w_q^q = 0$, which is a contradiction of the assumption $\int_\Omega f w^{p^\star} > 0$.

Finally suppose that $\int_\Omega f w^{p^\star} < 0$. Then, applying P.L. Lions' concentration compactness lemma recalled above, one gets that there exists two bounded and nonnegative measures μ and ν on $\overline{\Omega}$, some countable set of points (x_i) in $\overline{\Omega}$, and some sequence of non-negative numbers (μ_i) and (ν_i), which satisfy, up to a subsequence

$$|\nabla w_q|_p^p \rightharpoonup \mu \geq |\nabla w|_p^p + \sum_i \mu_i \delta_{x_i} \tag{3.23}$$

$$|w_q|^q \rightharpoonup \nu = |w|^{p^\star} + \sum_i \nu_i \delta_{x_i}. \tag{3.24}$$

Passing to the limit in (3.22), in the equality $\int_\Omega f w_q^q = \frac{-1}{|u_q|_p^q}$, and using (3.23) and (3.24), one obtains

$$I_\lambda(w) \leq -\sum_i \mu_i,$$

$$\int_\Omega f w^{p^\star} + \sum_i \nu_i f(x_i) = 0.$$

On the other hand, using $\int_\Omega f w^{p^\star} < 0$, one has

$$\alpha_\lambda \left(-\int_\Omega f w^{p^\star} \right)^{\frac{p}{p^\star}} \leq I_\lambda(w) \leq -\sum_i \mu_i.$$

Hence,

$$\sum_i \mu_i \leq -\alpha_\lambda \left(\sum_i \nu_i f(x_i) \right)^{\frac{p}{p^\star}}$$

Finally

$$\sum_i \mu_i \leq -\alpha_\lambda \sum_i (\nu_i f(x_i))^{\frac{p}{p^\star}} \leq -\alpha_\lambda \sup |f|^{\frac{p}{p^\star}} K(N,p)^p \sum_i \mu_i \leq \delta \sum_i \mu_i$$

for some $\delta < 1$. One obtains that $\mu_i = 0$ and then $\nu_i = 0$, as well as $\int_\Omega f w^{p^\star} = 0$, which contradicts the assumption.

As a consequence the claim is proved i.e. (u_q) is bounded in L^p.

Furthermore, since

$$\alpha_{\lambda,q} \geq (\lambda_1 - \lambda) \int_\Omega |u_q|^p$$

the sequence $\alpha_{\lambda,q}$ is bounded too. Let us denote by $\bar{\alpha}$ the limit of a subsequence. Clearly $\bar{\alpha} \leq \alpha_\lambda$. Since (u_q), $(u_q \geq 0)$ is bounded, one may extract a subsequence such that $u_q \rightharpoonup u$ in $W^{1,p}$. Let us recall that u_q satisfies:

$$\begin{cases} -\Delta_p u_q + (g-\lambda) u_q^{p-1} = -\alpha_{\lambda,q} f u_q^{q-1}, \\ \int_\Omega f u_q^q = -1 \end{cases} \tag{3.25}$$

Let us denote by σ the weak limit of a subsequence in $L^{\frac{p}{(p-1)}}(\Omega)$ of $\sigma_q := |\nabla u_q|^{p-1} \nabla u_q$. Then, passing to the limit in equation (3.25) one gets $u \geq 0$ and

$$-\mathrm{div}(\sigma) + (g-\lambda) u^{p-1} = -\bar{\alpha} f u^{p^\star - 1}. \tag{3.26}$$

Using again P.L. Lions' concentration lemma, there exist two bounded and nonnegative measures μ and ν on $\bar{\Omega}$, some countable sets of points (x_i) in $\overline{\Omega}$, and some sequence of nonnegative numbers (μ_i) and (ν_i), which satisfy, up to a subsequence

$$|\nabla u_q|_p^p \rightharpoonup \mu \geq |\nabla u|_p^p + \sum_i \mu_i \delta_{x_i} \text{ tightly on } \overline{\Omega},$$

$$|u_q|^q \rightharpoonup \nu = |u|^{p^\star} + \sum_i \nu_i \delta_{x_i}, \text{ tightly on } \overline{\Omega}.$$

Let us multiply equation (3.25) (resp. equation (3.26)) by $u_q \varphi$ (resp. $u\varphi$), for a function φ in $\mathcal{D}(\overline{\Omega})$. One obtains

$$\int_\Omega |\nabla u_q|^p \varphi + \int_\Omega \sigma_q \cdot \nabla \varphi u_q + \int_\Omega (g-\lambda) u_q^p \varphi = -\alpha_{\lambda,q} \int_\Omega f u_q^q \varphi \tag{3.27}$$

and

$$\int_\Omega (\sigma \cdot \nabla u) \varphi + \int_\Omega (\sigma \cdot \nabla \varphi) u + \int_\Omega (g-\lambda) u^p \varphi = -\bar{\alpha} \int_\Omega f u^{p^\star} \varphi. \tag{3.28}$$

By passing to the limit in (3.27), one gets

$$\begin{aligned} &\int_\Omega \mu\varphi + \int_\Omega (\sigma \cdot \nabla \varphi) u + \int_\Omega (g-\lambda) u^p \varphi \\ &= -\bar{\alpha} (\int_\Omega f u^{p^\star} \varphi + \sum_i \nu_i f(x_i) \varphi(x_i)). \end{aligned} \tag{3.29}$$

Subtracting (3.28) from (3.29) one obtains

$$\int_\Omega (\mu - \sigma \cdot \nabla u)\varphi = -\bar{\alpha}\left(\sum_i \nu_i f(x_i)\varphi(x_i)\right). \tag{3.30}$$

Using Lebesgue decomposition of $\mu := \mu^{ac} + \mu^s$, where μ^{ac} is the absolutely continuous part of μ, one derives

$$|\nabla u|^p \le \mu^{ac} = \sigma \cdot \nabla u, \tag{3.31}$$

$$\sum_i \mu_i \delta_{x_i} \le \mu^s = -\bar{\alpha}\nu_i f(x_i)\delta_{x_i}. \tag{3.32}$$

Suppose first that x_i is such that $f(x_i) \le 0$, then $\mu_i = \nu_i = 0$.

On the other hand, passing to the limit in equation (3.27) and using lower semi-continuity one has

$$I_\lambda(u) \le \bar{\alpha} < 0.$$

If $\int_\Omega f u^{p^\star} = 0$ this contradicts the assumption $\lambda \in]\lambda_1, \lambda_1 + \lambda^\star_{p^\star}[$. If $\int_\Omega f u^{p^\star} > 0$ one also gets a contradiction, since

$$0 \le \mu_\lambda \left(\int_\Omega f u^{p^\star}\right)^{\frac{p}{p^\star}} \le I_\lambda(u).$$

Suppose that $\int_\Omega f u^{p^\star} < 0$, then using (3.31) and (3.28) one has

$$\alpha_\lambda \left(-\int_\Omega f u^{p^\star}\right)^{\frac{p}{p^\star}} \le I_\lambda(u) \le -\bar{\alpha}\int_\Omega f u^{p^\star} \le -\alpha_\lambda \int_\Omega f u^{p^\star}.$$

From this, one obtains that $-\int_\Omega f u^{p^\star} \le 1$.

On the other hand the identity

$$\int_\Omega f u^{p^\star} + \sum_i \nu_i f(x_i) = -1$$

yields to $\sum_i \nu_i f(x_i) \le 0$, and since we are in the case $f(x_i) \ge 0$ we get $\nu_i f(x_i) = 0$ for all i. Using (3.32) one obtains that $\int_\Omega f u^{p^\star} = -1$ and $\mu_i = 0$. We have then

$$\alpha_\lambda \le \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)u^p \le \int_\Omega \sigma.\nabla u + \int_\Omega (g-\lambda)u^p = \bar{\alpha} \le \alpha_\lambda$$

which implies that $\bar{\alpha} = \alpha_\lambda$, $\sigma.\nabla u = |\nabla u|^p$, the convergence of ∇u_q is strong in $W^{1,p}(\Omega)$ and α_λ is achieved.

Second part. Since $\lim_{\lambda\to\lambda_1} \alpha_\lambda = \alpha_{\lambda_1} = 0$, one can choose λ sufficiently close to λ_1 in order to have

$$\alpha_\lambda > -\left(\sup |f|^{\frac{p}{p^\star}} K(N,p)^p\right).$$

Now let u_q be a function for which $\mu_{\lambda,q}$ is achieved, $u_q \geq 0$.

Claim. (u_q) is bounded in L^p when q goes to $p^\star$.

Suppose on the contrary that $|u_q|_p$ tends to infinity. Then, defining $w_q = \dfrac{u_q}{|u_q|_p}$, one obtains that w_q tends, up to a subsequence, to a function $w \in W_0^{1,p}(\Omega)$, $w \geq 0$ which satisfies $|w|_p = 1$, and

$$\int_\Omega |\nabla w|^p + \sum_i \mu_i + \int_\Omega (g-\lambda)w^p \leq 0,$$

$$\int_\Omega f w^{p^\star} + \sum_i \nu_i f(x_i) = 0$$

where (μ_i) and (ν_i) are as in the first part.

Suppose first that $\int_\Omega f w^{p^\star} = 0$. Then one gets a contradiction with the conditions on λ since

$$\int_\Omega |\nabla w|^p + \int_\Omega (g-\lambda)w^p \leq 0.$$

Suppose that $\int_\Omega f w^{p^\star} > 0$. Then by the definition of μ_λ one would obtain that

$$\mu_\lambda (\int_\Omega f w^{p^\star})^{\frac{p}{p^\star}} \leq |\nabla w|^p + \int_\Omega (g-\lambda)w^p \leq 0$$

Since $\mu_\lambda \geq 0$, this may happen only if $\mu_\lambda = 0$, and in the same time $I_\lambda(w) = 0$. Then, coming back to the previous inequalities, one has

$$I_\lambda(w) = 0 \leq \underline{\lim}_{q \to p^\star} I_\lambda(w_q) \leq 0$$

hence $I_\lambda(w_q) \to I_\lambda(w)$, and strong convergence holds. This implies that $\int_\Omega f w^{p^\star} = \lim_{q\to p^\star} \int_\Omega f w_q^q = 0$, which contradicts the assumption $\int_\Omega f w^{p^\star} > 0$.

Suppose finally that $\int_\Omega f w^{p^\star} < 0$, then one can write

$$\alpha_\lambda \left(- \int_\Omega f w^{p^\star} \right)^{\frac{p}{p^\star}} \leq \int_\Omega |\nabla w|_p^p + \int_\Omega (g-\lambda)w^p \leq -\sum_i \mu_i$$

and then

$$\begin{aligned}
\sum_i \mu_i &\leq (-\alpha_\lambda) \left(\sum_i \nu_i |f(x_i)| \right)^{\frac{p}{p^\star}} \\
&\leq (-\alpha_\lambda) \left(\sum_i \nu_i^{\frac{p}{p^\star}} |f(x_i)|^{\frac{p}{p^\star}} \right) \\
&\leq (-\alpha_\lambda) \sup |f|^{\frac{p}{p^\star}} \sum_i \mu_i K(N,p)^p \\
&\leq \delta \sum_i \mu_i
\end{aligned}$$

for some $\delta < 1$. Finally one has $\mu_i = 0$ for all i and then $\nu_i = 0$. Then $\int_\Omega f w^{p^\star} = 0$ which is absurd, as we remarked before. We have obtained that (u_q) is bounded. This proves the claim.

Let $\beta = \frac{1}{2}\left(K(N,p)^{-p} \sup |f|^{\frac{-p}{p^\star}} - \mu_{\lambda_1}\right)$ and suppose that λ is sufficiently close to λ_1 in order to ensure that

$$|\alpha_\lambda| < \beta.$$

Let (u_q) be a sequence of nonnegative minimizers for $\mu_{\lambda,q}$, $u_q \geq 0$. Then

$$-\Delta_p u_q + (g-\lambda)u_q^{p-1} = \mu_{\lambda,q} f u_q^{q-1} \tag{3.33}$$

$$\int_\Omega f u_q^q = 1.$$

By the previous computations, the sequence (u_q) is bounded in L^p, and since $(\mu_{\lambda,q})$ is bounded too, (u_q) is in fact bounded in $W^{1,p}$. Let us extract from it a subsequence such that

$$u_q \rightharpoonup u$$

in $W^{1,p}$ weakly. Let us denote by γ the limit of some subsequence of $\mu_{\lambda,q}$. One has $\gamma \leq \mu_\lambda \leq \mu_{\lambda_1}$.

Acting as we did in the first part, one gets

$$-\mathrm{div}(\sigma) + (g-\lambda)u^{p-1} = \gamma f u^{p^\star - 1}, \tag{3.34}$$

denoting by σ a weak limit of $|\nabla u_q|^{p-1}\nabla u_q$ in $L^{\frac{p}{p-1}}(\Omega)$.

Multiplying equation (3.33) (respectively (3.34)) by $u_q\varphi$ (respectively by $u\varphi$) with $\varphi \in \mathcal{D}(\overline{\Omega})$ and integrating over Ω, introducing measures μ and ν as in the concentration compactness lemma one gets

$$\mu^{ac} - \sigma.\nabla u = 0$$

$$\sum_i \mu_i \delta_i \leq \mu^s = \gamma \sum_i \nu_i f(x_i)\delta_i. \tag{3.35}$$

This last identity yields that γ cannot be zero: if it was, one would have $\mu_i = 0$, hence $\nu_i = 0$, and in the same time,

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)u^p = 0$$

and

$$\int_\Omega f u^{p^\star} = 1.$$

This is impossible, since for example, one has supposed that λ is not an eigenvalue. Then $\gamma > 0$. Moreover, if x_i is such that $f(x_i) < 0$, then $\mu_i = 0$, and so is ν_i. Since one has

$$|\nabla u|^p \leq \mu^{ac} = \sigma.\nabla u,$$

coming back to (3.34), one gets

$$\int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)u^p \le \int_\Omega \sigma\cdot\nabla u + \int_\Omega (g-\lambda)u^p = \gamma \int_\Omega f u^{p^\star}.$$

On another hand the identity

$$\int_\Omega f u^{p^\star} + \sum_i \nu_i f(x_i) = 1$$

implies that $\sum_i \nu_i f(x_i) \le 1$ if $\int_\Omega f u^{p^\star} \ge 0$. Suppose now that $\int_\Omega f u^{p^\star} < 0$. Then $\nu_f = \sum_i \nu_i f(x_i) > 1$. In the same time one has

$$\alpha_\lambda \left(-\int_\Omega f u^{p^\star}\right)^{\frac{p}{p^\star}} \le \int_\Omega |\nabla u|^p + \int (g-\lambda)u^p \le \gamma \int_\Omega f u^{p^\star}$$

and then

$$\nu_f \le 1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}.$$

As seen before if $f(x_i) < 0$, $\mu_i = 0$, hence $\nu_i = 0$. If $f(x_i) \ge 0$, the previous calculations imply that for all i, $\nu_i f(x_i) \le 1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}$. Finally

$$\begin{aligned}
\mu_i &\le \gamma \left(\frac{\nu_i f(x_i)}{1+\left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}}\right)\left(1+\left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}\right)\\
&\le \gamma \left(\frac{\nu_i f(x_i)}{1+\left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}}\right)^{1-\frac{p}{p^\star}} \left(\frac{\nu_i f(x_i)}{1+\left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}}\right)^{\frac{p}{p^\star}} \left(1+\left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}\right)\\
&\le \gamma \left(1+\left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^\star}}}\right)^{1-\frac{p}{p^\star}} K(N,p)^p \sup |f|^{\frac{p}{p^\star}} \mu_i\\
&\le \gamma\left(1+\frac{-\alpha}{\gamma}\right) K(N,p)^p \sup |f|^{\frac{p}{p^\star}} \mu_i\\
&\le K(N,p)^p \sup |f|^{\frac{p}{p^\star}} \mu_i(\gamma-\alpha)\\
&\le \delta\mu_i
\end{aligned} \tag{3.36}$$

for some $\delta < 1$. As a consequence $\mu_i = 0$ and then $\nu_i = 0$. Finally

$$\int_\Omega f u^{p^\star} = 1,$$

$$\mu_\lambda \le \int_\Omega |\nabla u|^p + \int_\Omega (g-\lambda)|u|^p \le \int_\Omega \sigma.\nabla u + \int_\Omega (g-\lambda)|u|^p \le \gamma$$

hence $\mu_\lambda = \gamma$, $|\nabla u|^p = \sigma.\nabla u = \mu$, the convergence is strong, and u is a minimizer for μ_λ. □

Remark 3.4 We have also obtained that $\mu_\lambda > 0$.

Corollary 3.5 *Suppose that $\int_\Omega f\phi^{p^\star} < 0$ and that there exists a minimizer for $\lambda = \lambda_1$, then there exist at least two minimizers for $\lambda > \lambda_1$, and λ sufficiently close to λ_1.*

Proof. Suppose that there exists a minimizer u_1 for the problem with $\lambda = \lambda_1$. Then

$$\begin{aligned}\inf_{\{u\in W_0^{1,p}(\Omega),\ \int_\Omega f|u|^{p^\star}=1\}} \left\{\int_\Omega |\nabla u|_p^p + \int_\Omega (g-\lambda)u^p\right\} &\le I_\lambda(u_1) < I_{\lambda_1}(u_1)\\ &= \inf I_{\lambda_1}(u)\\ &\le \frac{1}{K(N,p)^p \sup f(x)^{\frac{p}{p^\star}}}.\end{aligned}$$

As a consequence, using Theorem 1.6 one obtains that I_λ has a minimizer. □

4. Estimates and test functions

Let $x_0 \in \mathbf{R}^N$ and $r = |x - x_0|$ the euclidean distance from x_0 to x. For $p > 1$ given, p real such that $p < N$, we define the function u_ϵ by

$$u_\epsilon(x) = \left(\epsilon + r^{p/p-1}\right)^{1-N/p}$$

and the function v_ϵ by

$$v_\epsilon(x) = \left(\epsilon + r^{p/p-1}\right)^{1-N/p}\phi(r)$$

where $\phi : \mathbf{R} \to \mathbf{R}$, nonnegative and smooth, is such that $\phi(r) = 1$ for $r \le \delta/4$ and $\phi(r) = 0$ for $r \ge \delta$, $\delta > 0$ small. Recall here that

$$u_1(x) = \left(1 + r^{p/p-1}\right)^{1-N/p}$$

realizes the best constant for the embedding of $W^{1,p}(\mathbf{R}^N)$ in $L^{p^\star}(\mathbf{R}^N)$. Let also a and f be smooth functions defined in a neighborhood Ω of x_0. We assume in what follows that $f > 0$ in $B_{x_0}(\delta)$, and that $B_{x_0}(\delta) \subset \Omega$. For $u \in W_0^{1,p}(\Omega)$, we set

$$I(u) = \frac{\int_\Omega |\nabla u|^p dx + \int_\Omega (g(x) - \lambda_1)|u|^p dx}{\left(\int_\Omega f(x)|u|^{p^\star} dx\right)^{\frac{p}{p^\star}}}.$$

We also introduce

$$k_g = 0 \text{ if } g(x_0) < \lambda_1$$

$$k_g = \inf\{j \in \mathbf{N}, /\ j \ge 1 \text{ and } \Delta^j g(x_0) < 0\} \text{ if not}$$

$$k_f = \inf\{j \in \mathbf{N}^\star, /\Delta^j f(x_0) < 0\}$$

with the convention that $k_g = +\infty$ (resp. $k_f = +\infty$) if the corresponding set above is empty. Here $\Delta^j = \Delta^{j-1} \circ \Delta$, $j \geq 1$, where Δ is the usual Laplacian. When $N > p^2$, we define as in [10], [6]

$$k = \sup\{m \in \mathbf{N} / N > p^2 + 2m(p-1)\}$$

and for j integer, we set

$$\alpha_{N,j} = \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{1}{2})^{N-1}(2j+N)}{\Gamma(j+\frac{N}{2}+1)}$$

and

$$\tilde{\alpha}_j^{p,N} = \frac{\alpha_{N,j}}{(2j)!}\int_0^\infty \frac{r^{N+2j-1}dr}{\left(1+r^{\frac{p}{(p-1)}}\right)^{N-p}}$$

$$\tilde{\beta}_j^{p,N} = \frac{\alpha_{N,j}}{(2j)!}\frac{(N-p)^p}{(p-1)^{p-1}}\int_0^\infty \frac{r^{N+2j-1}dr}{\left(1+r^{\frac{p}{(p-1)}}\right)^{N}}.$$

Note that $\tilde{\alpha}_j^{p,N}$ exists as soon as $N > p^2 + 2j(p-1)$, that $\tilde{\beta}_j^{p,N}$ exists as soon as $N > 2j(p-1)$. One can find the explicit values of $\tilde{\alpha}_j^{p,N}$, $\tilde{\beta}_j^{p,N}$ in [10], Lemma 7.

Proposition 4.1 *Suppose that* $1 < p^2 < N$*and that* f *and* g *are* $\mathcal{C}^\infty(\overline{\Omega})$*. For* $\epsilon > 0$ *sufficiently small,*

$$I(v_\epsilon) < \frac{1}{K(N,p)^p f(x_0)^{\frac{p}{p^\star}}}$$

in each of the following cases

1. $k \geq k_g$*,* $k_f > k_g + \frac{p}{2}$*, and* $\Delta^{k_g}(g(x_0) - \lambda_1) < 0$.
2. $k \geq k_g$*,* $k_f < k_g + \frac{p}{2}$*, and* $\Delta^{k_f} f(x_0) > 0$.
3. $k \geq k_g$*,* $k_f = k_g + \frac{p}{2}$*, and* $\tilde{\alpha}_{k_g}^{p,n}(\Delta^{k_g}(g(x_0)-\lambda_1)f(x_0) - \tilde{\beta}_{k_f}^{p,n}\Delta^{k_f} f(x_0) < 0$
4. $k \leq k_g$*,* $k_f \leq k + \frac{p}{2}$*, and* $\Delta^{k_f} f(x_0) > 0$.

For example, the following corollary presents particular situations which enclose the results in the case where $p = 2$ obtained in [6], see also [1] in the case $p = 2$ and $g = 0$:

Corollary 4.2 *Suppose that* $1 < p^2 < n$*. For* $\epsilon > 0$ *small, one has that*

$$I(v_\epsilon) < \frac{1}{K(N,p)^p f(x_0)^{1-\frac{p}{N}}}$$

in each of the following situations

1. $1 < p < 2$ *and* $g(x_0) < \lambda_1$.
2. $p = 2$ *and* $\frac{8(N-1)}{(N-2)(N-4)}(g(x_0) - \lambda_1)f(x_0) - \Delta f(x_0) < 0$.
3. $p > 2$ *and* $g(x_0) = \lambda_1$*,* $\Delta g(x_0) = \Delta f(x_0) = 0$ *and* $\Delta^2 f(x_0) > 0$.

As a consequence of Proposition 4.1 One obtains that if f achieves its supremum on an interior point x_0 such that one of the situations described in 1. 2. 3. 4. occurs, then, there exists a solution to equation 1.1 for $\lambda = \lambda_1$ and for λ close to λ_1.

We do not give the proofs of Proposition 4.1 and Corollary 4.2 , because they are very technical and are already written in [10], in the coercive case. One must just replace in [10] the function a by the function $g - \lambda_1$.

5. Appendix

As mentioned in the introduction, in this appendix we want to prove the following

Proposition 2.1 *Suppose that* $f(x,t) = a(x)|t|^{q-2}t + b(x)|t|^{p-2}t$ *with* $1 < p < q$, *and* a *and* b *two continuous and bounded functions on* Ω. *Suppose that* $\bar{u}$ *is a weak supersolution for* $-\Delta_p u + f(x,u)$ $\bar{u} = 0$ *on* $\partial\Omega$, *and that* $\underline{u}$ *is a weak subsolution with* $\underline{u} = 0$ *on* $\partial\Omega$. *Suppose that there exists some constant* c *and* C *such that*

$$-\infty < c \leq \underline{u} \leq \bar{u} \leq C < +\infty$$

Then, there exists a solution u *between* $\underline{u}$ *and* $\bar{u}$

Proof. We follow the method of E. Hebey in [15].

Let k be choosen in order that the function

$$H(x,t) = f(x,t) + k|t|^{p-2}t$$

be increasing on $[\inf_{x\in\overline{\Omega}} \underline{u}, \sup_{x\in\overline{\Omega}} \overline{u}]$. Let u_1 be the solution of the variational problem

$$\inf_{u\in W_0^{1,p}(\Omega)} \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{k}{p}\int_\Omega |u|^p - \int_\Omega H(x,\overline{u})u.$$

The solution u_1 is unique and satisfies the following partial differential equation

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 = H(x,\overline{u})$$

and in particular

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 \leq -\Delta_p \overline{u} + k|\overline{u}|^{p-2}\overline{u}$$

and by the comparison principle one gets that $u_1 \leq \overline{u}$. On the other hand by the monotonicity properties of H

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 = H(x,\overline{u}) \geq H(x,\underline{u} \geq -\Delta_p \underline{u} + k|\underline{u}|^{p-2}\underline{u}$$

and then

$$u_1 \geq \underline{u}.$$

Finally u_1 is a supersolution since

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 = H(x,\overline{u}) \geq H(x,u_1),$$

hence

$$\underline{u} \le u_1 \le \overline{u}.$$

Iterating this process, one obtains the existence of a decreasing sequence u_n of supersolutions and

$$\underline{u} \le u_n \le \overline{u},$$

with

$$-\Delta_p u_n + k|u_n|^{p-2}u_n = H(x, u_{n-1}).$$

The sequence is, then, simply convergent and furthermore u_n is bounded in $W^{1,p}$ since it is bounded in L^∞ and

$$\int_\Omega |\nabla u_n|^p + k\int_\Omega |u_n|^p - \int_\Omega H(x, u_{n-1})u_n$$
$$\le \int_\Omega |\nabla \overline{u}|^p + k\int_\Omega |\overline{u}|^p - \int_\Omega H(x, \overline{u})u.$$

Extracting from it a subsequence one gets that there exists u such that $u_n \rightharpoonup u$ in $W^{1,p}$ weakly. Let σ be a weak limit of $|\nabla u_n|^{p-2}\nabla u_n$ in $L^{p'}$. It satisfies

$$-\text{div}\sigma + k|u|^{p-2}u = H(x, u).$$

Multiplying this by u and integrating by parts one gets

$$\int_\Omega \nabla u.\sigma + k\int_\Omega |u|^p = \int_\Omega H(x, u)u.$$

and on another hand passing to the limit in the equation satisfied by u_n, multiplied by u_n, one has

$$\lim \int_\Omega |\nabla u_n|^p + k\int_\Omega |u|^p = \int_\Omega H(x, u)u.$$

We have obtained that

$$\int_\Omega \sigma.\nabla u = \lim \int_\Omega |\nabla u_n|^p.$$

By using lower semicontinuity for the weak topology,

$$|\int_\Omega \sigma.\nabla u| \le \lim \left(\int_\Omega |\nabla u_n|^p\right)^{\frac{p}{p-1}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}}$$

and then

$$\lim \int_\Omega |\nabla u_n|^p \le \lim \left(\int_\Omega |\nabla u_n|^p\right)^{\frac{p}{p-1}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}}$$

hence

$$\lim \left(\int_\Omega |\nabla u_n|^p\right)^{\frac{1}{p}} \le \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}.$$

Since the other inequality is always true, one obtains that the convergence is strong, $\sigma = |\nabla u|^{p-2}\nabla u$, and u is a solution.

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