Eigenfunctions for singular fully nonlinear equations in unbounded regular domain

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1 Introduction

In this paper we prove the existence of a generalized eigenvalue and a corresponding eigenfunction for fully nonlinear operators singular or degenerate, homogeneous of degree $1+\alpha$, $\alpha > -1$ in non bounded domains of R^N. The main tool will be the Harnack's inequality. The key hypothesis on the operator, homogeneity (H1) and ellipticity (H2) are given later.

Very recently Davila, Felmer and Quaas [14, 15] proved Harnack's inequality in all dimensions N but in the singular case i.e. $\alpha < 0$. Here, in the two dimensional case, we prove Harnack's inequality for any $\alpha > -1$. The proof uses in an essential way this dimensional restriction. It follows the lines of the original proof of Serrin [25] in the linear case. For Harnack's inequalities in quasi-linear cases see [26] and [27]. Very recently C. Imbert [17] has proved an Harnack's inequality for fully nonlinear degenerate elliptic operators; let us mention that the class of operators he considers does not include those treated in this paper (see also [16] for degenerate elliptic equations in divergence form).

It is well known that Harnack's inequality allows to control the oscillations of the solutions and hence to prove uniform Hölder's estimates. It has been generalized to many 'weak' and nonlinear context, we are in particular thinking of those due to Krylov and Safonov for "strong solutions" [21], or the result of Caffarelli, Cabré [12] for fully nonlinear equations that are uniformly elliptic. Let us mention that in previous works on singular or degenerate fully nonlinear operators [4, 5] we proved Hölder's regularity of the solutions of Dirichlet problems in bounded domains. There the proof relied on the regularity of the solution on the boundary and the supremum of the solution. Hence in unbounded domains that tool cannot be used.

In the case treated here of fully nonlinear operators homogenous of degree $1+\alpha$, the Harnack inequality, due to Davila, Felmer and Quaas [14], is the following

Suppose that F does not depend on x and satisfies

$(H1)$ and (H2) as defined later and that $-1 < \alpha \leq 0$. Suppose that b, c and f are continuous and that u is a nonnegative solution of

$$F(\nabla u, D^2u) + b(x) \cdot \nabla u|\nabla u|^\alpha + cu^{1+\alpha} = f$$

in Ω. Then for all $\Omega' \subset\subset \Omega$ there exists some constant C which depends on a, A, α, b, c, N, Ω', Ω, such that

$$\sup_{\Omega'} u \leq C(\inf_{\Omega'} u + ||f||_{L^N(\Omega')}^{\frac{1}{1+\alpha}}).$$

Among all the consequences of Harnack's inequality, Berestycki, Nirenberg and Varadhan in their acclaimed paper [1] proved the existence of an eigenfunction for a linear, uniformly elliptic operator when no regularity of the boundary of the domain is known. The idea being that, close to the boundary, the solutions are controlled by the maximum principle in "small" domains, and, in the interior, one can use Harnack's inequality.

As it is well known, inspired by [1], the concept of eigenvalue in the case of bounded regular domains has lately been extended to fully nonlinear operators (see [7], [24], [4, 5], [18]). Two "principal eigenvalues" can be defined as the extremum of the values for which the maximum principle or respectively the minimum principle holds.

In this article we want to use the Harnack's inequality obtained here and in [14, 15] to study the eigenvalue problem in unbounded domains. Let us recall that in general, even for the Laplacian operator, the maximum principle does not hold in unbounded domain, hence we cannot define the "principal" eigenvalue in the same way as in the case of bounded domains. In [10] and [11] Capuzzo Dolcetta, Leoni and Vitolo study the conditions on the domain Ω in order for the Maximum principle to hold for fully nonlinear operators, extending the result of Cabré [9].

Furthermore let us mention that in unbounded domains there are several definitions that allow to construct different "eigenvalues" as the reader can see in Berestycki and Rossi [2] even for the Laplacian case. Here we define the first eigenvalue as the infimum of the first eigenvalues for bounded smooth domains included in Ω. We prove the existence of a positive eigenfunction for this so called eigenvalue, using Harnack's inequality.

We shall also prove the existence of solutions for equations below the eigenvalues. Observe that differently from the case of bounded domain, we can't use the maximum principle since in general it won't hold, hence again the Harnack's inequality will play a key role.

2 Assumptions on F

The following hypothesis will be considered, for $\alpha > -1$:

(H1) F is continuous on $\Omega \times \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$, and $\forall t \in \mathbb{R}^\star$, $\mu \geq 0$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.

(H2) For $p \in \mathbb{R}^N \setminus \{0\}$, $M \in S$, $N \in S$, $N \geq 0$

$$a|p|^\alpha tr(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha tr(N). \tag{2.1}$$

(H3) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$F(x, \zeta(x - y), X) - F(y, \zeta(x - y), -Y) \leq \omega(\zeta|x - y|^2).$$

Observe that when F is independent of x, condition (H3) is automatically satisfied.

Remark 2.1 *When no ambiguity arises we shall sometimes write $F[u]$ to signify $F(x, \nabla u, D^2u)$.*

Recall that examples of operators satisfying these conditions include the p-Laplacian with $\alpha = p - 2$ and

$$F(\nabla u, D^2u) = |\nabla u|^\alpha \mathcal{M}^\pm_{a,A}(D^2u)$$

where $\mathcal{M}^+_{a,A}$ is the Pucci operator $\mathcal{M}^+_{a,A}(M) = A\text{Tr}(M^+) - a\text{Tr}(M^-)$ and $\mathcal{M}^-_{a,A}(M) = a\text{Tr}(M^+) - A\text{Tr}(M^-)$.

For another example let $\alpha \leq 0$, B be some matrix with Lipschitz coefficients, and invertible for all $x \in \Omega$. Let us consider $A(x) = B^\star B(x)$ and the operator

$F(x, p, M) = |p|^\alpha (tr(A(x)(M))$, Then F satisfies (H1),.., (H3) arguing as in [3], example 2.4.

We assume that h and V are some continuous bounded functions on $\bar{\Omega}$ and

(H4) - Either $\alpha \leq 0$ and h is Hölder continuous of exponent $1 + \alpha$,

- or $\alpha > 0$ and

$$(h(x) - h(y)) \cdot (x - y) \leq 0$$

The solutions that we consider will be taken in the sense of viscosity, see e.g. [3] for precise definitions, let us recall that in particular we do not test when the gradient of the test function is null .

3 Main results

3.1 The Harnack's inequality in the two dimensional case.

In this subsection we state the Harnack's inequalities that will be proved in section 5 and used in section 4, together with some important corollary.

Theorem 3.1 (Harnack's inequality) *Suppose that Ω is a bounded domain in $\mathbb{R}^2$, and that F satisfies (H1) to (H3), h satisfies (H4).*

Let u be a positive solution of

$$F(x, \nabla u, D^2 u) + h(x).\nabla u |\nabla u|^\alpha + V(x) u^{1+\alpha} = 0 \quad in\ \Omega. \tag{3.1}$$

Let $\Omega' \subset\subset \Omega$. Then there exists $K = K(\Omega, \Omega', A, a, |h|_\infty, |V|_\infty)$ such that

$$\sup_{\Omega'} u \leq K \inf_{\Omega'} u. \tag{3.2}$$

Theorem 3.2 (Harnack's inequality) *Under the same hypothesis of Theorem 3.1, for f a bounded continuous function on Ω, let u be a positive solution of*

$$F(x, \nabla u, D^2 u) + h(x).\nabla u |\nabla u|^\alpha + V(x) u^{1+\alpha} = f(x) \quad in\ \Omega. \tag{3.3}$$

with V continuous, bounded and $V \leq 0$. Let $\Omega' \subset\subset \Omega$. Then there exists $K = K(\Omega, \Omega', A, a, |h|_\infty, |V|_\infty)$ such that

$$\sup_{\Omega'} u \leq K \left(\inf_{\Omega'} u + |f|_{L^\infty(\Omega)}^{\frac{1}{1+\alpha}} \right). \tag{3.4}$$

Corollary 3.3 *Let u be a solution of (3.3). Let R_o be such that $B(0, R_o) \subset \Omega$. Then there exists K which depend only on $A, a, |h|_\infty$ and R_o, such that for any $R < R_o$:*

$$\sup_{B(0,R)} u \leq K(\inf_{B(0,R)} u + R^{\frac{2+\alpha}{1+\alpha}} |f|_{L^\infty(B(0,R_o))}^{\frac{1}{1+\alpha}}). \tag{3.5}$$

As a consequence, for any solution u of (3.3), for any $\Omega' \subset\subset \Omega$ there exists $\beta \in (0,1)$ depending on the Harnack's constant in (3.5) such that $u \in C^{o,\beta}(\Omega')$.

An immediate consequence of Harnack's inequality is the following Liouville type result :

Corollary 3.4 (Liouville) *Let u be a solution of $F(x, \nabla u, D^2 u) = 0$ in $\mathbb{R}^2$, if u is bounded from below, then $u \equiv cte$.*

See [13] for other Liouville results.

3.2 Existence's results in unbounded domains.

Before stating the results in unbounded domains we recall what we mean by first eigenvalue and the property of these eigenvalues in the bounded case.

When Ω is a bounded domain we define

$$\overline{\lambda}(\Omega) = \sup\{\lambda, \exists\ \varphi > 0 \text{ in } \Omega, F[\varphi] + h(x) \cdot \nabla\varphi |\nabla\varphi|^\alpha + (V(x) + \lambda)\varphi^{1+\alpha} \leq 0\}$$

and

$$\underline{\lambda}(\Omega) = \sup\{\lambda, \exists\ \varphi < 0 \text{ in } \Omega, F[\varphi] + h(x) \cdot \nabla\varphi |\nabla\varphi|^\alpha + (V(x) + \lambda)\varphi |\varphi|^\alpha \geq 0\}.$$

We proved in [3] that there exists $\varphi > 0$ and $\psi < 0$ in Ω which are respectively a solution of

$$\begin{cases} F(x, \nabla\varphi(x), D^2\varphi(x)) + h(x) \cdot \nabla\varphi |\nabla\varphi|^\alpha + (V(x) + \overline{\lambda}(\Omega))\varphi^{1+\alpha} = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} F(x, \nabla\psi(x), D^2\psi(x)) + h(x)\cdot\nabla\psi|\nabla\psi|^\alpha + (V(x) + \underline{\lambda}(\Omega))|\psi|^\alpha\psi = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover φ and ψ are Hölder continuous.

When $\Omega \subset \mathbb{R}^N$ is not assumed to be bounded and $\mathcal{C}^2$, we define

$$\overline{\lambda}(\Omega) = \inf\{\overline{\lambda}(A), \text{for all smooth bounded domain } A, A \subset \Omega\},$$

and

$$\underline{\lambda}(\Omega) = \inf\{\underline{\lambda}(A), \text{for all smooth bounded domain } A, A \subset \Omega\}.$$

When no ambiguity arises we shall omit to write the dependence of the eigenvalues with respect to the set Ω.

We start by giving some lower bounds on the eigenvalues. For simplicity this will be done for $h \equiv 0$, $V \equiv 0$. If Ω is bounded it is easy to see that $\overline{\lambda}(\Omega) > 0$, while it is obvious that for $\Omega = \mathbb{R}^N$, $\overline{\lambda}(\Omega) = 0$. We wish to prove that this is not the case for all unbounded domains, in fact we shall see that, as long as Ω is bounded in one direction, then $\overline{\lambda}(\Omega) > 0$.

Proposition 3.5 *Suppose that Ω is contained in a strip of width M i.e. up to translation and rotation*

$$\Omega \subset [0, M] \times \mathbb{R}^{N-1}$$

then there exists $C = C(\alpha, a, A) > 0$ such that

$$\overline{\lambda}(\Omega) \geq \frac{C}{M^{2+\alpha}}. \tag{3.6}$$

Proof: Fix $\gamma \in (0,1)$ and observe that $u(x) = \sin^\gamma(x_1\frac{\pi}{4M} + \frac{\pi}{8}) \geq 0$ in Ω and

$$\begin{aligned} F[u] \leq\ & \gamma^{\alpha+1}\left(\frac{\pi}{4M}\right)^{\alpha+2} \sin^{\gamma(\alpha+1)-(\alpha+2)}(x_1\frac{\pi}{4M} + \frac{\pi}{8})\left(\cos(x_1\frac{\pi}{4M} + \frac{\pi}{8})\right)^\alpha \cdot \\ & a[\gamma - 1 - \gamma\sin^2(x_1\frac{\pi}{4M} + \frac{\pi}{8})]. \end{aligned}$$

Hence, using

$$\frac{3\pi}{8} \geq x_1\frac{\pi}{4M} + \frac{\pi}{8} > \frac{\pi}{8},$$

we get that there exists $C = C(\gamma, a, \alpha)$

$$F[u] + \frac{C}{M^{2+\alpha}} u^{\alpha+1} \leq 0.$$

Clearly this implies that $\overline{\lambda}(A) \geq \frac{C}{M^{2+\alpha}}$ for any $A \subset \Omega$. This gives (3.6) and it ends the proof.

In the next theorem we want to be in the same hypothesis for which Harnack's inequality holds, hence we consider the following condition:
(C) *F satisfies (H1), (H2); if $N \geq 3$ F is independent of x and $-1 < \alpha \leq 0$; if $N = 2$, $\alpha > -1$, F may depend on x and it satisfies (H3) .*

Theorem 3.6 *Suppose that Ω is some smooth domain possibly non bounded, of $\mathbb{R}^N$. Suppose that F satisfies (C), that h satisfies (H4), and that V is continuous, and bounded. Then there exist some functions $\phi > 0$ and $\psi < 0$ which are continuous and satisfy, respectively*

$$F[\phi] + h(x) \cdot \nabla\phi|\nabla\phi|^\alpha + (\overline{\lambda}(\Omega) + V(x))\phi^{1+\alpha} = 0 \text{ in } \Omega,$$

$$F[\psi] + h(x) \cdot \nabla\psi|\nabla\psi|^\alpha + (\underline{\lambda}(\Omega) + V(x))|\psi|^\alpha\psi = 0 \text{ in } \Omega.$$

Furthermore ϕ and ψ are Hölder continuous.

In the next proposition we treat existence of solutions below the eigenvalues.

Proposition 3.7 *For any $\lambda < \overline{\lambda}(\Omega)$, for any $f \in \mathcal{C}_c(\Omega)$ non positive, there exists $v > 0$ solution of*

$$F[v] + h(x) \cdot \nabla v|\nabla v|^\alpha + (\lambda + V(x))v^{1+\alpha} = f \textit{ in } \Omega.$$

Furthermore, for $f \not\equiv 0$ there exists C, which depends on the support of f, such that

$$|v|_\infty \leq C|f|_\infty^{\frac{1}{1+\alpha}}.$$

Similarly if $\lambda < \underline{\lambda}(\Omega)$, for any $f \in \mathcal{C}_c(\Omega) \geq 0$, there exists $v < 0$ solution of

$$F[v] + h(x) \cdot \nabla v|\nabla v|^\alpha + (\lambda + V(x))|v|^\alpha v = f \textit{ in } \Omega.$$

Remark 3.8 *As mentioned in the introduction, in [4] we proved some Hölder's regularity result for all $\beta \in [0, 1[$ in bounded regular domains, see Proposition 4.3, but for homogeneous or regular boundary conditions. More precisely the Hölder's constants depend on the L^∞ norm of u and u is zero on the boundary.*

From this we derive some Hölder's uniform estimates for sequences of solutions and this implies that a subsequence of such solutions converges for a subsequence towards a solution. This cannot be used in the proof of the results above, indeed we shall need compactness results inside bounded sets Ω_n whose size increases, for sequence of functions which have uniform L^∞ bounds on bounded fixed sets, but for which the $L^\infty(\Omega_n)$ norm may go to infinity.

4 Known results.

We now recall the following weak comparison principle which will be used for the proof of Theorem 3.2.

Theorem 4.1 *Suppose that F, h and V are as above and that $V \leq 0$.*
Suppose that f and g are continuous and bounded and that u and v satisfy

$$\begin{array}{rcl} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + V(x)|u|^\alpha u & \geq & g \quad in \quad \Omega \\ F(x, \nabla v, D^2 v) + h(x) \cdot \nabla v |\nabla v|^\alpha + V(x)|v|^\alpha v & \leq & f \quad in \quad \Omega \\ u \leq v & & \quad on \quad \partial\Omega. \end{array}$$

Suppose that $f < g$, then $u \leq v$ in Ω. Moreover if $V < 0$ and $f \leq g$ the result still holds.

We shall also need for the proof of Theorem 3.1 another comparison principle :

Theorem 4.2 *Suppose that $\tau < \overline{\lambda}(\Omega)$, $f \leq 0$, f is upper semi-continuous and g is lower semi-continuous with $f \leq g$.*
Suppose that there exist u continuous and $v \geq 0$ and continuous, satisfying

$$\begin{array}{rcl} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + (V(x) + \tau)|u|^\alpha u & \geq & g \quad in \quad \Omega \\ F(x, \nabla v, D^2 v) + h(x) \cdot \nabla v |\nabla v|^\alpha + (V(x) + \tau) v^{1+\alpha} & \leq & f \quad in \quad \Omega \\ u \leq v & & \quad on \quad \partial\Omega. \end{array}$$

Then $u \leq v$ in Ω in each of these two cases:
1) If $v > 0$ on $\overline{\Omega}$ and either $f < 0$ in Ω, or $g(\bar{x}) > 0$ on every point $\bar{x}$ such that $f(\bar{x}) = 0$.
2) If $v > 0$ in Ω, $f < 0$ in $\overline{\Omega}$ and $f < g$ on $\overline{\Omega}$.

The proof can be found in [3]. We also recall some regularity results

Proposition 4.3 *Suppose that F satisfies (H1),(H2), (H3). Let f be some continuous function in $\overline{\Omega}$. Let u be a viscosity non-negative bounded solution of*

$$\begin{cases} F(x, \nabla u, D^2u) + h(x) \cdot \nabla u|\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Then, if h is continuous and bounded, for any $\gamma < 1$ there exists some constant C which depends only on $|f|_\infty$, $|h|_\infty$ and $|u|_\infty$ such that :

$$|u(x) - u(y)| \leq C|x - y|^\gamma$$

for any $(x, y) \in \overline{\Omega}^2$.

Under slightly stronger condition on F we also prove the Lipschitz regularity of the solutions.

5 Proofs of the Main results

5.1 Proof of existence

We start by proving the existence results:

Proof of Theorem 3.6. We shall only explicitly write the proof of the existence of $\phi > 0$, the case of $\psi < 0$ being analogous. Let Ω_n be a sequence of bounded subsets such that

$$\Omega_n \subset\subset \Omega_{n+1} \subset\subset \Omega, \quad \overline{\lambda}(\Omega_n) \to \overline{\lambda}(\Omega) \quad \text{and} \quad \cup_n \Omega_n = \Omega.$$

Let f_n be a sequence of functions in $\mathcal{C}_c(\Omega_n \setminus \overline{\Omega_{n-1}})$, $f_n \leq 0$ and not identically zero. Since $\overline{\lambda}(\Omega_n) > \overline{\lambda}(\Omega)$, for all n there exists $u_n \geq 0$ which solves

$$\begin{cases} F[u_n] + h(x) \cdot \nabla u_n|\nabla u_n|^\alpha + (\overline{\lambda}(\Omega) + V(x))u_n^{1+\alpha} = f_n & \text{in } \Omega_n \\ u_n = 0 & \text{on } \partial\Omega_n. \end{cases}$$

Let $x_0 \in \Omega_1$, then $u_n(x_0) > 0$ for all n by the strict maximum principle. Define

$$v_n(x) = \frac{u_n(x)}{u_n(x_0)}$$

that we extend by zero outside Ω_n, obtaining in such a way a continuous function. Let O', O be bounded regular subdomains, $O \subset\subset O'$. We prove that

$(v_n)_n$ converges uniformly on $K' = \overline{O}'$. Indeed there exists N_0 such that Ω_n contains K' for all $n \geq N_0$. As a consequence on O' , for $n \geq N_0$

$$F[v_n] + h(x) \cdot \nabla v_n |\nabla v_n|^\alpha + (\overline{\lambda}(\Omega) + V(x)) v_n^{1+\alpha} = 0 \quad \text{in } O'.$$

Moreover $v_n(x_0) = 1$. Using Harnack's inequality of Theorem 3.1 we know that there exists some constant $C_{K'}$ such that

$$\sup v_n \leq C_{K'} (\inf v_n) \leq C_{K'}.$$

This implies in particular that v_n is bounded independently of n in K'.

By taking $f_n = -V(x) v_n^{1+\alpha}$ in Corollary 3.3 on the open set O', one gets that $(v_n)_n$ is relatively compact in O. A subsequence of $(v_n)_n$ will converge to a solution ϕ of

$$F[\phi] + h(x) \cdot \nabla \phi |\nabla \phi|^\alpha + (\overline{\lambda}(\Omega) + V(x)) \phi^{1+\alpha} = 0 \quad \text{in} \quad O.$$

$\phi(x_0) = \lim v_n(x_0) = 1$ implies that ϕ cannot be identically zero. By strict maximum principle on compacts sets of Ω, $\phi > 0$ inside Ω. Since O can be taken arbitrarily large this ends the proof.

Proof of Proposition 3.7. We consider only the case $f \leq 0$ and $\lambda < \overline{\lambda}(\Omega)$. We first treat the case $f \not\equiv 0$. Let K be the compact support of $f \leq 0$. As in the previous proof let Ω_n be a sequence of bounded sets such that

$$\Omega_n \subset \Omega_{n+1} \text{ and } \cup_n \Omega_n = \Omega.$$

Let u_n be a (positive) solution of

$$\begin{cases} F[u_n] + h(x) \cdot \nabla u_n |\nabla u_n|^\alpha + (V(x) + \lambda) u_n^{1+\alpha} = f & \text{in } \Omega_n \\ u_n = 0 & \text{on } \partial\Omega_n. \end{cases}$$

Let φ^+ be given in Theorem 3.6 such that

$$F[\varphi^+] + h(x) \cdot \nabla \varphi^+ |\nabla \varphi^+|^\alpha + (\overline{\lambda}(\Omega) + V(x)) {\varphi^+}^{1+\alpha} = 0$$

with L^∞ norm 1 in K.

Rescaling φ^+,

$$\varphi_1 = \frac{\varphi^+ \sup |f|^{\frac{1}{1+\alpha}}}{(\overline{\lambda} - \lambda)^{\frac{1}{1+\alpha}} \inf_K \varphi^+},$$

by homogeneity is a solution of

$$F[\varphi_1] + h(x)\cdot\nabla\varphi_1|\nabla\varphi_1|^\alpha + (\lambda + V(x))\varphi_1^{1+\alpha} = (\lambda - \overline{\lambda})\frac{(\varphi^+)^{1+\alpha}\sup(-f)}{(\overline{\lambda}-\lambda)(\inf_K \varphi^+)^{1+\alpha}} \le f.$$

We can apply the comparison principle Theorem 4.2 in Ω_n, since $\varphi_1 > 0$ on $\partial\Omega_n$, to derive that

$$0 \le u_n \le \varphi_1$$

for any n. Using the same argument as in the proof of Theorem 3.6, on every compact subset of Ω there is a subsequence of $(u_n)_n$ converging to u, a solution of

$$F[u] + h(x)\cdot\nabla u|\nabla u|^\alpha + (V(x)+\lambda)u^{1+\alpha} = f \quad \text{in } \Omega.$$

By the strict maximum principle applied on bounded sets of Ω we get that $u > 0$.

We now prove the case $f \equiv 0$. Without loss of generality we only treat the case $\lambda < \overline{\lambda}(\Omega)$.

Let Ω_n be a sequence of bounded sets such that

$$\Omega_n \subset \Omega_{n+1} \text{ and } \cup_n \Omega_n = \Omega.$$

Let u_n be a solution of

$$\begin{cases} F[u_n] + h(x)\cdot\nabla u_n|\nabla u_n|^\alpha + (V(x)+\lambda)u_n|u_n|^\alpha = 0 & \text{in } \Omega_n \\ u_n = 1 & \text{on } \partial\Omega_n. \end{cases}$$

Since $\lambda < \inf\{\overline{\lambda}(\Omega_n)\}$, u_n exists, is well defined and $u_n > 0$ in Ω_n. Let $x_0 \in \Omega_1$.

Rescaling u_n we get that $v_n = \frac{u_n}{u_n(x_0)}$ is a solution of

$$F[v_n] + h(x)\cdot\nabla v_n|\nabla v_n|^\alpha + (V(x)+\lambda)v_n^{1+\alpha} = 0.$$

By Harnack's inequality, for every relatively compact domain O, $(v_n)_n$ is bounded on $K = \overline{O}$.

Using the compactness results on O there exists a subsequence v_n which converges uniformly to some v solution of

$$F[v] + h(x)\cdot\nabla v|\nabla v|^\alpha + (V(x)+\lambda)v^{1+\alpha} = 0.$$

Moreover, since $v_n(x_0) = 1$, and the convergence is uniform one gets that $v(x_0) = 1$, hence v is not identically zero and by the strict maximum principle $v > 0$ in Ω.

5.2 Proofs of Harnack's inequality in the two dimensional case.

The proofs that we propose follow the lines in Gilbarg Trudinger [19] and Serrin [25], with some new arguments that make explicite use of the eigenfunction in bounded domains. This extends the result of [14] to the case $\alpha > 0$, but only in the two dimensional case.

In the proof of Theorems 3.1 and 3.2 we shall use the following lemma

Lemma 5.1 *Suppose that F, h and V are as above. Let b and c, be some positive parameters, $x_o = (x_{o1}, x_{o2}) \in \mathbb{R}^2$. Let*

$$E = \{x = (x_1, x_2),\ \sigma^2(x) := \frac{(x_1 - x_{o1})^2}{b^2} + \frac{(x_2 - x_{o2})^2}{c^2} \le 1,\ x_1 - x_{o1} > \frac{b}{2}\}.$$

Then there exists a constant $\gamma > 0$ such that

$$v(x) = \frac{e^{-\gamma\sigma^2(x)} - e^{-\gamma}}{e^{-\gamma/4} - e^{-\gamma}},$$

satisfies in E

$$F(x, \nabla v, D^2 v) - |h|_\infty |\nabla v|^{1+\alpha} - |V|_\infty v^{1+\alpha} > 0. \tag{5.1}$$

(Note that v is strictly positive inside E and is zero on the elliptic part of the boundary).

Remark 5.2 *The same result holds for the symmetric part of ellipsis : $E = \{x = (x_1, x_2),\ \sigma^2(x) \le 1,\ x_1 - x_{o1} < \frac{-b}{2}\}$.*

Proof of Lemma 5.1.

Without loss of generality one can assume that $x_o = 0$.

Let $\tilde{v} = \frac{e^{-\gamma\sigma^2}}{e^{-\gamma/4} - e^{-\gamma}}$ and let B be the diagonal 2×2 matrix such that $B_{11} = \frac{1}{b^2}$ and $B_{22} = \frac{1}{c^2}$. Then $\nabla v = -2\gamma\tilde{v}Bx$ and

$$D^2 v = (2\gamma)(2\gamma Bx \otimes Bx - B)\tilde{v}.$$

Since B and $Bx \otimes Bx$ are both nonnegative,

$$a(tr(D^2v)^+) - A(tr(D^2v)^-) \ge \left(a\gamma^2 4(\frac{x_1^2}{b^4} + \frac{x_2^2}{c^4}) - 2(A + a)\gamma(\frac{1}{b^2} + \frac{1}{c^2})\right)\tilde{v}.$$

We define

$$m = \inf\left(b^{-\alpha}, 2^{\alpha}(\frac{1}{b^2}+\frac{1}{c^2})^{\alpha/2}\right) \text{ and } M = 2^{1+\alpha}(\frac{1}{b^2}+\frac{1}{c^2})^{\frac{1+\alpha}{2}}.$$

We choose

$$\gamma = \sup\left(\frac{4(A+a)}{a}(1+\frac{b^2}{c^2}), \frac{4|h|_\infty M b^2}{ma}, \left(\frac{4|V|_\infty b^2}{am}\right)^{\frac{1}{2+\alpha}}\right). \tag{5.2}$$

Using (H1):

$$\begin{aligned} & F(x,\nabla v, D^2 v) + h(x)\cdot \nabla v|\nabla v|^\alpha + V(x)v^{1+\alpha} \geq \\ \geq\ & |\nabla v|^\alpha (a(tr(D^2v)^+) - A(tr(D^2v)^-)) - |h|_\infty |\nabla v|^{1+\alpha} - |V|_\infty v^{1+\alpha} > 0. \end{aligned}$$

This ends the proof of Lemma 5.1.

Remark 5.3 *The proof in the case $f \not\equiv 0$ follows the lines of the case $f \equiv 0$ but the ellipsis are rescaled. Hence we shall use, for ρ_o to be defined, $\sigma(\frac{x}{\rho_o})$ instead of σ. It will be important to observe that γ does not depend on bounded ρ_o. This is immediate from the definition of γ in (5.2) and the constants m, M, b and c involved.*

Proof of Theorem 3.1:

Let us remark that the existence of a positive solution u implies in particular that $\overline{\lambda}(\Omega) \geq 0$. Moreover without loss of generality we can suppose that $\overline{\lambda}(\Omega) > 0$. Indeed, by the properties of the eigenvalue there exists $\Omega_1 \subset \Omega$ such that $\Omega' \subset\subset \Omega_1$ and $\overline{\lambda}(\Omega_1) > \lambda(\Omega) \geq 0$. Then we consider the proof in Ω_1 instead of Ω.

We shall prove the following claims :

Claim 1: Suppose that $\Omega = B(0,1)$. For any $P \in B(0,\frac{1}{3})$ there exists K which depends only on a, A, and bounds on h and V such that

$$u(P) \geq K u(0).$$

Claim 2: For any $P \in B_{\frac{1}{4}}(0)$, there exist K_1 and K_2 such that

$$K_1 u(0) \leq u(P) \leq K_2 u(0).$$

Claim 3: Suppose that $\Omega = B(0,R)$. For any $P \in B(0,\frac{R}{4})$ such that

$$K_1 u(0) \leq u(P) \leq K_2 u(0),$$

where K_1 and K_2 depend on R only when h and V are not identically 0.
Claim 4: The inequality holds true for Ω bounded and $\Omega' \subset\subset \Omega$.

Proof of Claim 1 :

So we are in the case $\Omega = B(0,1)$ with $\overline{\lambda}(B(0,1)) > 0$. Hence there exists $\delta > 0$ sufficiently small such that $\overline{\lambda}(B(0,1+\delta)) > 0$ as well.

Let u_δ be the corresponding positive eigenfunction such that u_δ has the L^∞ norm equals to $\frac{1}{2}$, i.e. u_δ satisfies

$$\begin{cases} F[u_\delta] + h(x)\cdot \nabla u_\delta |\nabla u_\delta|^\alpha + \left(V(x) + \overline{\lambda}(B(0,1+\delta))\right) u_\delta^{1+\alpha} = 0 \text{ in } B(0,1+\delta) \\ u_\delta = 0 \text{ on } \quad \partial B(0,1+\delta). \end{cases}$$

Let $\chi = u(0)u_\delta$.

Let $G_1 = \{x \in B(0,1), u(x) > \chi(x)\}$. The connected component of G_1, denoted G, which contains 0, contains at least one point on $\partial B(0,1)$. Indeed, if not, $\overline{G}$ would be included in $B(0,1)$, hence by the comparison principle in G since $u(x) = \chi$ on ∂G and and since $0 < \overline{\lambda}(B(0,1+\delta))$, χ is a supersolution of $F[\chi]+h(x).\nabla\chi|\nabla\chi|^\alpha+(V(x))\chi^{1+\alpha} < 0$, then applying the comparison Theorem 4.2 in the set G, one would get $u(x) \leq \chi$ in G, a contradiction. Without loss of generality we will suppose that the boundary point has coordinates $(0,1)$.

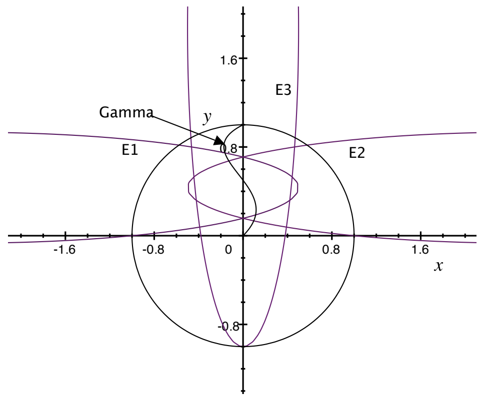
We now introduce the part of ellipsis E_i $i = 1,2,3$ given by:

$$E_1 = \{(x_1,x_2),\ \frac{(x_1+\frac{5}{2})^2}{9} + 4(x_2 - \frac{\sqrt{3}}{4})^2 \leq 1,\ x_1 \geq -1\}$$
$$E_2 = \{(x_1,x_2),\ \frac{(x_1-\frac{5}{2})^2}{9} + 4(x_2 - \frac{\sqrt{3}}{4})^2 \leq 1,\ x_1 \leq 1\}.$$

Observe that the segment $[-1/2,1/2] \times \{\frac{\sqrt{3}}{4}\}$ is contained in $E_1 \cap E_2$. while $(0,0) \notin E_1 \cap E_2 \subset B(0,1)$.

The third part of ellipse E_3 has its straight part in $E_1 \cap E_2$ and vertex at $(0,-1)$:

$$E_3 = \{(x_1,x_2),\ 4x_1^2 + \left(\frac{x_2 - 1 - \frac{\sqrt{3}}{2}}{2+\frac{\sqrt{3}}{2}}\right)^2 \leq 1, x_2 \leq \sqrt{3}/4\}.$$



Let v_i be given by Lemma 5.1, such that $0 \le v_i \le 1$, $v_i = 0$ on the elliptic boundary of E_i and v_i satisfies $F[v_i] + h(x) \cdot \nabla v_i |\nabla v_i|^\alpha + V(x) v_i^{\alpha+1} > 0$.

There exists Γ some simple and regular curve which is included in G and links $(0,0)$ to $(0,1)$.

Let $E = E_1 \cup E_2$. We denote by ∂E^+ and ∂E^- the superior and inferior boundary of E. Necessarily Γ cuts ∂E^+ and ∂E^-. Suppose that φ parametrizes Γ with φ in $\mathcal{C}^2$, $\varphi(0) = (0,0)$ and $\varphi(1) = (0,1)$. Let $t^- = \sup\{t, \varphi(t) \in \partial E^-\}$ and $t^+ = \inf\{t, \varphi(t) \in \partial E^+\}$, and let $p^- = \varphi(t^-)$, $p^+ = \varphi(t^+)$. The portion of curve (p^-, p^+) in Γ is such that for all $t \in]t^-, t^+[$, $\varphi(t)$ is in the interior of E. Using the orientation of the portion of curve between p^- and p^+ one gets that this portion of curve separates E in two parts, the left E_l and the right E_r.

Let $z \in E_1 \cap E_2$; if $z \in E_l$, we choose $D = E_2 \cap E_l$, otherwise $z \in E_r$ and $D = E_1 \cap E_r$. In the first case D has a boundary made of parts of ∂E_2 and the arc $\widehat{p^-, p^+} \cap E_2$. In the second one the boundary of D has a boundary made of parts of E_1 and $\widehat{p^-, p^+} \cap E_1$.

For example if we denote by $\kappa := \inf_{B(0,1)} u_\delta > 0$, in the second case we have obtained:

$$u - \kappa u(0) v_1 > \kappa u(0)(1 - v_1) > 0 \text{ on } \Gamma \cap E_1$$

$$u - \kappa u(0) v_1 = u > 0 \text{ on } \partial E_1$$

and analogous inequalities in the first case.

Using the comparison principle in Theorem 4.2, we have obtained that

$$u(P) \ge \kappa u(0) \min\{v_1(P), v_2(P)\} \quad \text{for all} \quad P \in E_1 \cap E_2.$$

Now we will use this to prove a similar inequality in E_3.

One has :

$$u \geq \inf_{\{P\in\partial E_3, x_2=\sqrt{3}/4\}} \min(v_1(P), v_2(P)) v_3 \kappa u(0).$$

Indeed, this inequality holds on ∂E_3, because on the elliptic part of E_3, $v_3 = 0$ and the straight part is included in $E_1 \cap E_2$, where the inequality holds. Now by the comparison principle (Theorem 4.2) the inequality holds true in E_3.

We apply this in the ball $B(0, 1/3)$ which is strictly included in the interior of E_3; defining

$$m_3 = \inf_{B(0,1/3)} v_3,$$

we have obtained that

$$\begin{aligned} u &\geq \kappa u(0) \inf_{\{P\in\partial E_3, x_2=\sqrt{3}/4\}} (\min(v_1(P), v_2(P))) m_3 \\ &\geq u(0)\kappa \inf_{\{P\in\partial E_3, x_2=\sqrt{3}/4\}} (\min(v_1(P), v_2(P))) m_3. \end{aligned}$$

Proof of Claim 2. Fix any point $\overline{P}$ in $B_{\frac{1}{4}}(0)$. Then

$$B_{\frac{3}{4}}(\overline{P}) \subset B_1(0), \text{ and } 0 \in B_{\frac{1}{4}}(\overline{P}).$$

Hence by Claim 1 we have that

$$u(\overline{P}) \leq K u(0)$$

but always by Claim 1, $u(0) \leq K u(\overline{P})$. This ends the proof of Claim 2, by choosing $K_1 = \frac{1}{K}$ and $K_2 = K$.

Proof of Claim 3. Now $\Omega = B(0, R)$ and u is a positive solution of (3.1). Using the homogeneity of F, let $v(p) := u(Rp)$ satisfies

$$F(Rx, \nabla v, D^2 v) + Rh(Rx) \cdot \nabla v |\nabla v|^\alpha + R^{\alpha+2}(V(Rx)) v^{\alpha+1} \leq 0, \text{ in } B_1(0).$$

Hence we are in the conditions of the previous case with h replaced by $Rh(Rx)$ and $V(x)$ replaced by $R^{\alpha+2}V(Rx)$. We have obtained that v satisfies, for any $P \in B_{\frac{R}{3}}(0)$:

$$v(0) \leq K v(P) \text{ i.e. } u(0) \leq K u(Q) \text{ for } Q \in B_{\frac{R}{3}}(0).$$

Observe that K depends on γ (see (5.2)), but when $R \leq R_o$ it can be chosen independently on R.

Proof of Claim 4. This is standard potential theory procedure. Let K be a compact connected subset of Ω. And let $R = \inf\{r, d(P, \partial\Omega) \leq r, \text{ for any } P \in K\}$. Suppose that P and Q are any two points of K. Then there exists a continuous curve $\Gamma \subset K$ joining P and Q. We can find a finite number of points $P = P_1, P_2, \ldots, P_k = Q$ such that

$$P_i \in \Gamma, \ |P_i P_{i+1}| \leq \frac{R}{4}, \ B_R(P_i) \subset \Omega.$$

Hence applying the previous results, observing that

$$\overline{\lambda}(\Omega) < \overline{\lambda}(B_R(P_i))$$

we get

$$u(P) \leq K_2 u(P_2) \leq K_2^2 u(P_3) \leq K_2^k u(Q).$$

This ends the proof of Theorem 3.1.

Before giving the proof of Theorem 3.2, for convenience of the reader we state an elementary Lemma (without proof) which will be used later in the proof.

Lemma 5.4 *Let $B_o \in \mathbb{R}^+$ and $B_1 \in \mathbb{R}^+$, and $q = \frac{\alpha+2}{\alpha+1}$. For any x_o let $\tilde{\rho} = |x - x_o|$ and $w_1 = C_1 \tilde{\rho}^q$. Then for $\tilde{\rho} \leq \frac{aq^{2+\alpha} B_o}{2B_1}$ and $C_1 = \left(\frac{|f|_\infty 2}{B_o a q^{2+\alpha}}\right)^{\frac{1}{1+\alpha}}$, w_1 is a solution of*

$$B_o |\nabla w|^\alpha \mathcal{M}^-(D^2 w) - B_1 |\nabla w|^{1+\alpha} \geq |f|_\infty.$$

Proof of Theorem 3.2

The case $V(x) \equiv 0$. The proof proceeds with the same steps as in the case $f \equiv 0$, the difference being that instead of comparing u with the functions v_i defined in Lemma 5.1 we will need to compare it for some constant C with $Cv_i + w$ where w is a subsolution of (3.3), and since the operator is fully nonlinear and not sublinear, we need to prove that $Cv_i + w$ is a subsolution.

Recall that $q = \frac{\alpha+2}{\alpha+1}$.

First step: Let

$$\rho_o = \inf(1, \frac{aq^{2+\alpha}}{2|h|_\infty}), \quad \text{and} \quad C_1 = (\frac{2|f|_\infty}{aq^{2+\alpha}})^{\frac{1}{1+\alpha}}.$$

Using Lemma 5.4 the function $w_1 = C_1\rho^q$ satisfies, for $\rho \le \rho_o$,

$$F[-w_1] + h \cdot \nabla(-w_1)|\nabla w_1|^\alpha \le -|f|_\infty,$$

Of course $\frac{u(0)}{2} - w_1$ is also a supersolution.

Let $G_1 = \{x \in B(0,\rho_o), u(x) > \frac{u(0)}{2} - C_1\rho^q\}$. G_1 is an open set which contains 0. Let G be the connected component of G_1 which contains 0. By the comparison principle the boundary of G contains at least one point of $\partial B(0,\rho_o)$. One can assume that this point is $(0,\rho_o)$.

We now proceed to the second step. We introduce the half-ellipsis,

$$E_1 = \{(x_1,x_2),\ \sigma_1^2 := \frac{(x_1 + \frac{5\rho_o}{2})^2}{9\rho_o^2} + 4\frac{(x_2 - \frac{\rho_o\sqrt{3}}{4})^2}{\rho_o^2} \le 1,\ x_1 \ge -\rho_o\}$$

$$E_2 = \{(x_1,x_2),\ \sigma_2^2 := \frac{(x_1 - \frac{5\rho_o}{2})^2}{9\rho_o^2} + 4\frac{(x_2 - \frac{\rho_o\sqrt{3}}{4})^2}{\rho_o^2} \le 1,\ x_1 \le \rho_o\}$$

$$E_3 = \{(x_1,x_2),\ \sigma_3^2 := \frac{4x_1^2}{\rho_o^2} + \frac{\left(x_2 - \rho_o(1+\frac{\sqrt{3}}{2})\right)^2}{\rho_o^2(2+\frac{\sqrt{3}}{2})^2} \le 1, x_2 \le \frac{\sqrt{3}\rho_o}{4}\}.$$

Let $\tilde\rho^2 = x_1^2 + (x_2+3\rho_o)^2$ and $w = C_1\tilde\rho^q$. Let Γ be a regular curve which links 0 to $(0,\rho_o)$ and is included in G, then since $\tilde\rho > \rho$, one always has $u > \frac{u(0)}{2} - C_1\tilde\rho^q$ on Γ. As in the case $f \equiv 0$, E_l and E_r indicate respectively the left and right part of $E_1 \cap E_2$ with respect to Γ.

For $i = 1,2,3$, we want to prove that

Claim A: *It is possible to choose v_i as in Lemma 5.1 and C_1 and ρ_o such that*

$$\frac{u(0)}{2}v_i + w \le u + 2^{2+2q}\rho_o^q C_1 \ \text{on } \partial(E_l \cap E_1) \text{ and on } \partial(E_r \cap E_2)$$
$$F[\frac{u(0)}{2}v_i + w] + h(x)\cdot\nabla(\frac{u(0)}{2}v_i + w)|\nabla(\frac{u(0)}{2}v_i + w)|^\alpha \ge |f|_\infty;$$

in $E_1\cap E_2$ *for* $i=1,2$ *and for some* $\chi_1<1$

$$\frac{\chi_1 u(0)}{2}v_3+w\leq u+2^{2+2q}\rho_o^q C_1 \ \textit{on}\ \partial E_3$$

and

$$F[\frac{\chi_1 u(0)}{2}v_3+w]+h(x)\cdot\nabla(\frac{u(0)}{2}\chi_1 v_3+w)|\nabla(\frac{u(0)}{2}\chi_1 v_3+w)|^\alpha\geq|f|_\infty;$$

in E_3.

If this is done the conclusion follows from the comparison principle as in the case $f\equiv 0$. Indeed Claim A with $\chi_1=\inf_{\{x\in\partial E_3,\ x_2=\frac{\sqrt{3}}{4},i=1,2\}}(v_i)\leq 1$ gives that there exist some constant K and K' such that

$$u\geq Ku(0)-K'|f|_\infty^{\frac{1}{1+\alpha}}.$$

The proof of Claim A is different in the case $\alpha\leq 0$ and $\alpha>0$. We start by the case $\alpha\leq 0$. Recall that

$$v_i=\frac{e^{-\gamma_i\sigma_i^2}-e^{-\gamma_i}}{e^{\frac{-\gamma_i}{4}}-e^{-\gamma_i}}$$

where e.g.

$$\gamma_1=\sup\left(\frac{4(A+a)}{a}(1+36),\frac{4|h|_\infty M_1 9}{m_1 a}\right)$$

for

$$m_1=3^{-\alpha},\quad \text{and}\quad M_1=2^{1+\alpha}(\frac{1}{9}+4)^{\frac{1+\alpha}{2}}.$$

For the following we shall replace the constant γ_i by $\gamma\equiv\sup\gamma_i$ which is also convenient to our goal.

We need to observe that

$$|\nabla v_i|\leq\frac{4\gamma}{\rho_o}\tilde{v}$$

where $\tilde{v}=\frac{e^{-\gamma\sigma^2}}{e^{\frac{-\gamma}{4}}-e^{-\gamma}}$. Note that $v_o=\frac{e^{\frac{-\gamma}{4}}}{e^{\frac{-\gamma}{4}}-e^{-\gamma}}\geq\tilde{v}\geq e^{\frac{-3\gamma}{4}}v_o$. With all these choices of constants, the computation in Lemma 5.1 gives for $i=1,2,3$

$$|\nabla v_i|^\alpha\left(\mathcal{M}^-(D^2v_i)-h(x)\cdot\nabla v_i\right)\ \geq\ \frac{2^{2\alpha-1}\gamma^{2+\alpha}a\tilde{v}^{1+\alpha}}{9\rho_o^{2+\alpha}}$$

$$
\begin{aligned}
&\geq \frac{2^{2\alpha-1}\gamma^{2+\alpha}a(e^{\frac{-3\gamma}{4}}v_o)^{1+\alpha}}{9\rho_o^{2+\alpha}} \\
&:= \frac{c_2}{\rho_o^{\alpha+2}}.
\end{aligned}
$$

We now consider two cases :

Either $\frac{u(0)^{1+\alpha}c_2}{2^{1+\alpha}\rho_o^{\alpha+2}} > |f|_\infty$ and then it is enough to take $w = 0$ and Claim A is satisfied,

or

$$
\frac{u(0)^{1+\alpha}c_2}{2^{1+\alpha}\rho_o^{\alpha+2}} \leq |f|_\infty.
$$

Recall that $\tilde{\rho} = (x_1^2 + (x_2 + 3\rho_o)^2)^{\frac{1}{2}}$.

We choose $C_1 = \sup\left(\frac{e^{\frac{3\gamma}{4}}(18)^{\frac{1}{1+\alpha}}}{(a\gamma)^{\frac{1}{1+\alpha}}}, \left(\frac{2^{1-\alpha}}{aq^{2+\alpha}}\right)^{\frac{1}{1+\alpha}}\right)|f|_\infty^{\frac{1}{1+\alpha}}$

$$
w = C_1\tilde{\rho}^q.
$$

With this choice of C_1 we shall prove that $|\frac{u(0)}{2}\nabla v_i| \leq |\nabla w|$. For simplicity we shall do the computation only for v_1. Observe first that

$$
\begin{aligned}
|\frac{u(0)}{2}\nabla v_1| &\leq \frac{2u(0)\gamma\tilde{v}}{\rho_o} \\
&\leq 2(\frac{u(0)v_o\gamma}{\rho_o}) \\
&\leq e^{\frac{3\gamma}{4}}\left(\frac{2^3|f|_\infty\rho_o 9}{a\gamma}\right)^{\frac{1}{1+\alpha}} \\
&\leq C_1 q(2\rho_o)^{q-1} \leq qC_1\tilde{\rho}^{q-1}.
\end{aligned}
$$

From this, similar calculations and Lemma 5.4, we derive that, for $i = 1, 2, 3$,

$$
\begin{aligned}
(|\nabla\left(\frac{u(0)}{2}v_i\right)| + |\nabla w|)^\alpha\left(\mathcal{M}^-(D^2w) - |h|_\infty|\nabla w|\right) &\geq (2qC_1\tilde{\rho}^{q-1})^\alpha\frac{aq^2}{2}C_1\tilde{\rho}^{q-2} \\
&\geq 2^\alpha C_1^{1+\alpha}q^\alpha\frac{aq^2}{2} \\
&\geq |f|_\infty.
\end{aligned}
$$

Moreover from the choice of γ_i, one has

$$
\mathcal{M}^-(D^2v_i) - |h|_\infty|\nabla v_i| \geq 0,
$$

and, using the simple inequality $|X+Y|^\alpha \geq (|X|+|Y|)^\alpha$ this ends the proof of Claim A in the case $\alpha \leq 0$.

We now consider the case $\alpha > 0$. In order to prove Claim A, we need to evaluate $\nabla v_i \cdot \nabla w$; this is done in Lemma 5.5 below. Applying it, there exists some $1 > \delta > 0$ such that for $i = 1, 2$ one has in $E_1 \cap E_2$,

$$\begin{aligned} |\nabla \left(\frac{u(0)}{2} v_i\right) + \nabla w|^2 &\geq |\nabla \left(\frac{u(0)}{2} v_i\right)|^2 + |\nabla w|^2 + 2(-1+\delta^2)|\nabla \left(\frac{u(0)}{2} v_i\right)||\nabla w| \\ &\geq \delta^2 |\nabla \left(\frac{u(0)}{2} v_i\right)|^2 + \delta^2 |\nabla w|^2 \end{aligned}$$

and in E_3 denoting as χ the constant $\chi_1 = \inf_{\{P\in\partial E_3, x_2=\sqrt{3}/4\}} \min(v_1(P), v_2(P))$

$$|\nabla \left(\frac{u(0)}{2}\chi_1 v_3\right) + \nabla w|^2 \geq \delta^2 |\nabla \left(\frac{u(0)\chi_1}{2} v_3\right)|^2 + \delta^2 |\nabla w|^2.$$

Using Lemma 5.1 we can choose γ_i in order that v_i satisfies

$$\delta^\alpha 2^{\frac{-|\alpha-2|}{2}} |\nabla v_i|^\alpha \mathcal{M}^-_{a,A}(D^2 v_i) - 2^\alpha |h|_\infty |\nabla v_i|^{\alpha+1} \geq 0$$

in E_i, this gives, e.g. for $i = 1$,

$$\gamma_1 = \sup\left(\frac{4(A+a)}{a}(1+36), \frac{2^{\frac{|\alpha-2|}{2}+\alpha+2} 9|h|_\infty M_1}{\delta^\alpha m_1 a}\right)$$

for some obvious definitions of m_1 and M_1.

We now choose $\rho_o = \inf(\frac{2^{-\frac{|\alpha-2|}{2}-\alpha-3}\delta^\alpha a q^{2+\alpha}}{|h|_\infty}, 1)$, $C_1 = \left(\frac{2^{\frac{|\alpha-2|}{2}+2}|f|_\infty}{aq^{2+\alpha}\delta^\alpha}\right)^{\frac{1}{1+\alpha}}$, by Lemma 5.4, one gets

$$2^{\frac{-|\alpha-2|}{2}}\delta^\alpha |\nabla w|^\alpha \mathcal{M}^-(D^2 w) - 2^\alpha |h|_\infty |\nabla w|^{1+\alpha} \geq aq^{2+\alpha} 2^{\frac{-|\alpha-2|}{2}-1}\delta^\alpha C_1^{1+\alpha} \geq |f|_\infty.$$

This implies that for $i = 1, 2$, in E_i

$$\begin{aligned} &|\frac{u(0)}{2}\nabla v_i + \nabla w|^\alpha \mathcal{M}^-(\frac{u(0)}{2}D^2 v_i + D^2 w) - |\frac{u(0)}{2}\nabla v_i + \nabla w|^{\alpha+1}|h|_\infty \\ \geq\ & 2^{\frac{-|\alpha-2|}{2}}\delta^\alpha(|\frac{u(0)}{2}\nabla v_i|^\alpha + |\nabla w|^\alpha)\mathcal{M}^-(\frac{u(0)}{2}D^2 v_i + D^2 w) \\ &-2^\alpha|\frac{u(0)}{2}(\nabla v_i|^{\alpha+1} + |\nabla w|^{1+\alpha}) \\ \geq\ & 2^{-\frac{|\alpha-2|}{2}}\delta^\alpha|\frac{u(0)}{2}\nabla v_i|^\alpha \mathcal{M}^-(\frac{u(0)}{2}D^2 v_i) \\ &-|h|_\infty 2^\alpha|\frac{u(0)}{2}\nabla v_i|^{1+\alpha} + 2^{\frac{-|\alpha-2|}{2}}\delta^\alpha|\nabla w|^\alpha \mathcal{M}^-(D^2 w) - |h|_\infty 2^\alpha|\nabla w|^{1+\alpha} \geq |f|_\infty \end{aligned}$$

and also

$$\begin{aligned}
&|\frac{u(0)}{2}\chi_1\nabla v_3+\nabla w|^\alpha\mathcal{M}^-(\frac{u(0)}{2}\chi_1 D^2v_3+D^2w)-|\frac{u(0)}{2}\chi_1\nabla v_3+\nabla w|^{\alpha+1}|h|_\infty\\
\geq\ & 2^{\frac{-|\alpha-2|}{2}}\delta^\alpha(|\frac{u(0)}{2}\chi_1\nabla v_3|^\alpha\mathcal{M}^-(\frac{u(0)}{2}\chi_1 D^2v_3)\\
&-|h|_\infty 2^\alpha|\frac{u(0)}{2}\chi_1\nabla v_3|^{1+\alpha}+2^{\frac{-|\alpha-2|}{2}}\delta^\alpha|\nabla w|^\alpha\mathcal{M}^-(D^2w)-|h|_\infty 2^\alpha|\nabla w|^{1+\alpha}\\
\geq\ & |f|_\infty.
\end{aligned}$$

In order to complete the proof of Claim A we check the boundary conditions. Let $\tilde\rho_o=4\rho_o$ in order that in $\rho<\rho_o$, $\tilde\rho<\tilde\rho_o$. If $x\in E_l\cap E_1$ which is made of some part of ∂E_1 and some part of Γ, one gets that $u+2C_1\tilde\rho_o^q>w$ on the boundary of ∂E_1 since u is positive. On Γ it is true since $u\geq\frac{u(0)}{2}-C_1\tilde\rho^q$.

Lemma 5.5 *There exists $\delta\in[0,1[$ such that in $E_1\cap E_2$ for $i=1$ and $i=2$*

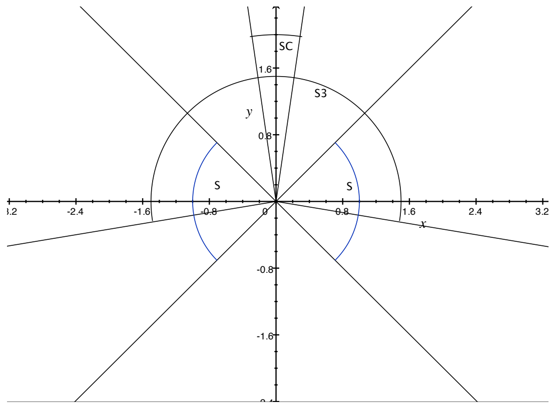
$$\langle\nabla v_i,\nabla w\rangle\geq(-1+\delta^2)|\nabla v_i||\nabla w|$$

and in E_3

$$\langle\nabla v_3,\nabla w\rangle\geq(-1+\delta^2)|\nabla v_3|\ |\nabla w|.$$

Proof For homogeneity reasons, we can assume that $\rho_o=1$. Then $\nabla v_i=\gamma_i B_i x\tilde v$, with $B_1x:=-(\frac{x_1+\frac{5}{2}}{9},4(x_2-\frac{\sqrt3}{4}))$, $B_2x:=-(\frac{x_1-\frac{5}{2}}{9},4(x_2-\frac{\sqrt3}{4}))$, and $B_3x=-(4x_1,\frac{x_2-1-\frac{\sqrt3}{2}}{(2+\frac{\sqrt3}{2})^2})$, while $\nabla w=C_1\tilde\rho^{q-2}(Cx)$ with $Cx=(x_1,x_2+3)$.

It is an elementary but tedious calculation to see that for $x\in E_1\cap E_2$ the vectors B_1x, B_2x lie in the circular sector S defined by $\frac{6\sqrt{11}}{5}|x_1|\geq x_2\geq\frac{-6\sqrt{11}}{5}|x_1|$, while Cx lies in a sector S_o defined by $\frac{\sqrt3+12}{2}|x_1|\leq x_2$. Hence if θ_1 is the angle between the sectors then the first equality is satisfied with $-1+\delta=\cos\theta_1$. Similarly for the second case.



The circles of smaller radius indicate the sectors spanned by $B_i x$ and the circle of larger radius indicates the sector spanned by Cx, as can be seen the angle between $B_i x$ and Cx is never π.

The case $V \leq 0$, $f \neq 0$.

We begin with the case $\alpha < 0$.

We now consider the case $V \leq 0$. We choose $\rho_o = \inf \left(\frac{aq^{2+\alpha}}{4|h|_\infty}, \left(\frac{aq^{2+\alpha}}{|V|_\infty 8} \right)^{\frac{1}{\alpha+2}} \right)$ and $C_1 = \left(\frac{2^{1-\alpha}}{aq^{2+\alpha}} \right)^{\frac{1}{1+\alpha}}$ and we choose the constant γ_i as in Lemma 5.1 with $V \leq 0$, in order that $|\nabla v_i|^\alpha \mathcal{M}^-(D^2 v_i) - |h|_\infty |\nabla v_i|^{1+\alpha} - |V|_\infty v_i^{1+\alpha} \geq 0$.

Let us note that since $V \leq 0$, u is also a subsolution of

$$|\nabla u|^\alpha \left(\mathcal{M}^-(D^2 u) - h(x) \cdot \nabla u \right) \geq f$$

and then the first step is still valid. We obtain that there exists some point on the boundary of $\partial B(0, \rho_o)$ such that $u \geq \frac{u(0)}{2} - C_1 \rho_o^q$. We can assume that this point is $(0, \rho_o)$.

We now consider as previously two cases:

-Either $\dfrac{u(0)^{1+\alpha} c_2}{2^{1+\alpha} \rho_o^{\alpha+2}} > |f|_\infty$ and then for $i = 1, 2$, $\frac{u(0) v_i}{2}$ is a subsolution of the equation in $E_1 \cap E_2$ and so is $\frac{u(0) \chi v_3}{2}$ in E_3 , while $u + 2C_1 \rho_o^q$ is a supersolution of the same equation. Moreover in $E_1 \cap E_2$, using the fact that the boundary is made of arcs of Γ or of parts of the boundary of E_i one gets that $u + 2C_1 (4\rho_o)^q \geq \frac{u(0)}{2} \inf(v_1, v_2)$ in $E_1 \cap E_2$. And now we do the final step as in the case where $f = 0$, i.e. we prove that $u + 2C_1 (4\rho_o)^q \geq \frac{u(0)}{2} \inf_{x_2 = \frac{\sqrt{3}}{4}, x \in \partial E_3} (v_1, v_2) v_3$.

- Or $\dfrac{u(0)^{1+\alpha}c_2}{2^{1+\alpha}\rho_o^{\alpha+2}} \leq |f|_\infty$.

In that case we define $C_1 = \sup\left(e^{\frac{3\gamma}{4}}\left(\frac{9}{a\gamma}\right)^{\frac{1}{1+\alpha}}, \left(\frac{2^{1-\alpha}}{aq^{2+\alpha}}\right)^{\frac{1}{1+\alpha}}, 2^{\frac{1}{1+\alpha}}c_2^{\frac{-1}{1+\alpha}}\right)|f|_\infty^{\frac{1}{1+\alpha}}$ and $\tilde{\rho}$ as in the case $V=0$ and we observe that $|\nabla v_i|\frac{u(0)}{2} \leq |\nabla w|$. Let us note that, with the choice of C_1 above,

$$\frac{v_i u(0)}{2} \leq w$$

and

$$2^\alpha|\nabla w|^\alpha\left(\mathcal{M}^-(D^2w) - |h|_\infty|\nabla w|\right) - 2^{\alpha+1}|V|_\infty w^{1+\alpha} \geq |f|_\infty.$$

We now write for $i = 1, 2$

$$\begin{aligned}
&(|\nabla\frac{u(0)v_i}{2}| + |\nabla w|)^\alpha\left(\mathcal{M}^-(D^2(\frac{u(0)v_i}{2}+w) - h(x)\cdot(\nabla\frac{u(0)v_i}{2}+\nabla w)\right)\\
- \ & |V|_\infty(\frac{u(0)v_i}{2}+w)^{1+\alpha}\\
\geq \ & (|\nabla\frac{u(0)v_i}{2}| + |\nabla w|)^\alpha\left(\mathcal{M}^-(D^2(\frac{u(0)v_i}{2})) - h(x)\cdot(\nabla\frac{u(0)v_i}{2})\right)\\
+ \ & (2|\nabla w|)^\alpha\left(\mathcal{M}^-(D^2(w) - h(x)\cdot\nabla w\right) - 2^{\alpha+1}|V|_\infty w^{1+\alpha}\\
\geq \ & 0 + |f|_\infty
\end{aligned}$$

and for $i = 3$ and $\chi_1 = \inf_{\{x_2=\frac{\sqrt{3}}{4}, x\in\partial E_3\}}\inf(v_1, v_2)$

$$\begin{aligned}
&(|\nabla\frac{u(0)\chi_1 v_3}{2}| + |\nabla w|)^\alpha\left(\mathcal{M}^-\left((D^2(\frac{u(0)\chi_1 v_3}{2}+w)\right) - h(x)\cdot(\nabla\frac{u(0)\chi_1 v_3}{2}+\nabla w)\right)\\
- \ & |V|_\infty(\frac{u(0)\chi_1 v_3}{2}+w)^{1+\alpha}\\
\geq \ & (|\nabla\frac{u(0)\chi_1 v_3}{2}| + |\nabla w|)^\alpha\left(\mathcal{M}^-(D^2(\frac{u(0)\chi_1 v_3}{2})) - h(x)\cdot(\nabla\frac{u(0)\chi_1 v_3}{2})\right)\\
+ \ & (2|\nabla w|)^\alpha\left(\mathcal{M}^-(D^2(w)) - h(x)\cdot\nabla w\right) - 2^{\alpha+1}|V|_\infty w^{1+\alpha}\\
\geq \ & |f|_\infty.
\end{aligned}$$

The rest of the proof is analogous to the one done in the previous cases, observing that, since $V \leq 0$, $u+2C_1(4\rho_o)^q$ is also a supersolution of the equation.

We now treat the case $\alpha > 0$. The notations B_i, C, δ are the same as in the case $V = 0$.

Since $V \leq 0$, u is also a subsolution of $F[u] + h(x) \cdot \nabla u |\nabla u|^\alpha \geq f$ so the first step is the same, more precisely if we choose $\rho_o < \inf \left(\left(\frac{|h|_\infty}{4|V|_\infty} \right)^{1+\alpha}, \frac{\delta^\alpha 2^{\frac{-|\alpha-2|}{2} - \alpha - 2} a q^2}{|h|_\infty} \right)$, and $C_1 = \left(\frac{2^{\frac{|\alpha-2|}{2}+3} |f|_\infty}{a q^{2+\alpha} \delta^\alpha} \right)^{\frac{1}{1+\alpha}}$, where δ is as in the proof of $V = 0$, $\alpha > 0$ and $f \neq 0$, $w_1 = -C_1 \rho^q$ is a supersolution of $F[w_1] + h(x) \cdot \nabla w_1 |\nabla w_1|^\alpha \leq -|f|_\infty$, then so is $\frac{u(0)}{2} + w_1$. We obtain always by the same reasoning that there exists some point on the boundary $\rho = \rho_o$ on which $u > \frac{u(0)}{2} - C_1 \rho^q$.

For the second step we must prove that one can chose v_i such that in E_i

$$\delta^\alpha 2^{\frac{-|\alpha-2|}{2}} \mathcal{M}^-(D^2 v_i) |\nabla v_i|^\alpha - 2^\alpha |h|_\infty |\nabla v_i|^{\alpha+1} - 2^\alpha |V|_\infty v_i^{1+\alpha} > 0.$$

This can be done by choosing γ_i such that

$$\gamma_i = \sup \left(\frac{4(A+a)}{a} (1 + \frac{b_i^2}{c_i^2}), \frac{2^{\frac{|\alpha-2|}{2}+\alpha+3} |h|_\infty M_i b_i^2}{\delta^\alpha m_i a}, \left(\frac{2^{\frac{|\alpha-2|}{2}+\alpha+2} |V|_\infty b_i^2}{a m_i \delta^\alpha} \right)^{\frac{1}{2+\alpha}} \right)$$

(with obvious definitions of b_i, c_i, M_i, m_i, on the model of the proof of Lemma 5.1).

Let $\tilde{\rho}$ be defined as in the previous proof, then $w = C_1 \tilde{\rho}^q$ is a solution of $2^{\frac{-|\alpha-2|}{2}} \delta^\alpha \mathcal{M}^-(D^2 w) - 2^\alpha |h(x)|_\infty |\nabla w|^{\alpha+1} - 2^\alpha |V|_\infty w^{1+\alpha} \geq |f|_\infty$, and then $\frac{u(0)}{2} v_i + w$ is for $i = 1, 2$ a sub-solution of

$$F[\frac{u(0)}{2} v_i + w] + h(x) \cdot \nabla (\frac{u(0)}{2} v_i + w) |\nabla (\frac{u(0)}{2} v_i + w)|^\alpha + V(x) (\frac{u(0)}{2} v_i + w)^{\alpha+1} \geq |f|_\infty$$

in $E_1 \cap E_2$ and

$$\begin{aligned} F[\frac{u(0)}{2} \chi_1 v_3 + w] &+ h(x) \cdot \nabla (\frac{u(0)}{2} \chi_1 v_3 + w) |\nabla (\frac{u(0)}{2} \chi_1 v_3 + w)|^\alpha + \\ &+ V(x) \left(\frac{u(0)}{2} \chi_1 v_3 + w \right)^{1+\alpha} \geq |f|_\infty \end{aligned}$$

in E_3.

We observe now that since $V \leq 0$, $u + 2C_1 \tilde{\rho}_o^q$ satisfies

$$F[u + 2C_1 \tilde{\rho}_o^q] + h(x) \cdot \nabla (u + 2C_1 \tilde{\rho}_o^q) |\nabla (u + 2C_1 \tilde{\rho}_o^q)|^\alpha + V(x) (u + 2C_1 \tilde{\rho}_o^q)^{1+\alpha} \leq f.$$

The rest of the proof is the same.

Proof of Corollary 3.3. Suppose that u is a solution in Ω which contains $B(0,R_o)$. Let v be defined as $v(x) =: u(Rx)$. Then v satisfies in $B(0,\frac{R_o}{R})$,

$$F(Rx,\nabla v,D^2v)(x)+Rh(Rx)\cdot\nabla v|\nabla v|^\alpha+R^{2+\alpha}V(Rx)v^{1+\alpha}=R^{2+\alpha}f(Rx)$$

Applying Harnack's inequality for v we get the desired result for u.

Let $R_o>0$ such that $B(x_o,4R_o)\subset\Omega'\subset\subset\Omega$. We define for any $R<R_o$

$$M_i=\max_{B_(x_o,iR)}u,\quad m_i=\min_{B_(x_o,iR)}u$$

for $i=1$ and $i=4$. Then $u-m_i$ is a solution of

$$F[u-m_i]+h(x)\nabla(u-m_i)|\nabla(u-m_i)|^\alpha=-V(x)u^{1+\alpha}$$

in $B(x_o,iR)$ and hence u satisfies

$$\sup_{B(x_o,R)}(u(x)-m_4)\le K\inf_{B(x_o,R)}(u(x)-m_4)+KR^{\frac{2+\alpha}{\alpha+1}}M_4|V|_\infty^{\frac{1}{1+\alpha}}$$

In the same way, using the operator $G(x,p,M)=-F(x,p,-M)$, and the function M_i-u, we get

$$G(x,\nabla u,D^2(M_i-u))+h(x)\cdot|\nabla(M_i-u)|^\alpha\nabla(M_i-u)\ =\ V(x)u^{1+\alpha}$$

in $B(0,iR)$. We get with some constant K which can be taken equal to the previous one

$$\sup_{B(x_o,R)}(M_4-u(x))\le K\inf_{B(x_o,R)}(M_4-u(x))+KR^{\frac{2+\alpha}{\alpha+1}}M_4|V|_\infty^{\frac{1}{1+\alpha}}.$$

Summing the inequalities we obtain for some constant K' independent of $R\le R_o$

$$M_1-m_1\le\frac{K-1}{K+1}(M_4-m_4)+K'R^{\frac{2+\alpha}{\alpha+1}}.$$

The rest of the proof is classical, just apply Lemma 8.23 in [19].

Proof of Corollary 3.4. Let $c_0=\inf_{\mathbb{R}^2}u$ and let $w=u-c_0$. Clearly w satisfies in $\mathbb{R}^2$:

$$F[w]=0,\ w\ge0,\ \inf w=0.$$

Suppose by contradiction that $w > 0$ somewhere, then applying the strong maximum principle one gets that $w > 0$ in the whole of $\mathbb{R}^2$.

By definition of the infimum, for any $\varepsilon > 0$ there exists $P \in \mathbb{R}^2$ such that $w(P) \leq \varepsilon$. Now for any $Q \in \mathbb{R}^2$ consider the ball centered at P and of radius $4|PQ|$, by Harnack's inequality and more precisely using Claim 4 in the proof, we get that

$$w(Q) \leq K_2 w(P) \leq K_2 \varepsilon.$$

Observe that K_2 doesn't depend on the distance $|PQ|$ because $h = V \equiv 0$, hence it doesn't depend on the choice of Q. Since this holds for any ε we get $w \equiv 0$.

References

[1] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains.* Comm. Pure Appl. Math. 47 (1994), no. 1, 47–92.

[2] H. Berestycki, L. Rossi, *On the principal eigenvalue of elliptic operators in* $\mathbb{R}^N$ *and applications.* J. Eur. Math. Soc. (JEMS) 8 (2006), no. 2, 195–215.

[3] I. Birindelli, F. Demengel, *Comparison principle and Liouville type results for singular fully nonlinear operators*, Ann. Fac. Sci Toulouse Math, (6)13 (2004), N.2, 261-287., 2004.

[4] I. Birindelli, F. Demengel, *Eigenvalue and Maximum principle for fully nonlinear singular operators* Advances in Partial Diff. Equations.**11**,1 (2006) 91-119.

[5] I. Birindelli, F. Demengel, *Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators,* Comm. Pure and Applied Analysis, 6 (2007),pp. 335-366.

[6] I. Birindelli, F. Demengel, *The Dirichlet problem for singular fully nonlinear operators*, Discrete and Cont. Dynamical Sys., (2007), Special vol. pp. 110-121

[7] J. Busca,M.J. Esteban, A. Quaas, *Nonlinear eigenvalues and bifurcation problems for Pucci's operator* , Ann. de l'Institut H. Poincaré, Analyse non linéaire, 22 (2005), pp. 187-206.

[8] J. Busca, B. Sirakov, *Harnack type estimates for nonlinear elliptic equations systems and applications*, Ann. Inst. H. Poincare, Anal. Nonl. 21 (5) (2004), 543-590.

[9] X. Cabré, *On the Alexandroff-Bakelman-Pucci estimate and the reversed Hlder inequality for solutions of elliptic and parabolic equations.* Comm. Pure Appl. Math. 48 (1995), no. 5, 539–570.

[10] I. Capuzzo-Dolcetta, F. Leoni, A. Vitolo, *The Alexandrov-Bakelman-Pucci weak maximum principle for fully nonlinear equations in unbounded domains.* Comm. Partial Differential Equations 30 (2005), no. 10-12, 1863–1881.

[11] I. Capuzzo Dolcetta, A. Vitolo, *A qualitative Phragmn-Lindelf theorem for fully nonlinear elliptic equations.* J. Differential Equations 243 (2007), no. 2, 578–592.

[12] L. Caffarelli, X. Cabré, Fully-nonlinear equations Colloquium Publications 43, American Mathematical Society, Providence, RI,1995.

[13] A. Cutrì, F. Leoni, *On the Liouville property for fully nonlinear equations.* Ann. Inst. H. Poincar Anal. Non Linaire 17 (2000), no. 2, 219–245.

[14] G. Davila, P. Felmer, A. Quaas *Harnack Inequality For Singular Fully Nonlinear Operators and some existence results* Preprint.

[15] G. Davila, P. Felmer, A. Quaas *Alexandroff -Bakelman-Pucci estimate for singular or degenerate fully nonlinear elliptic equations,* Comptes rendus de l'académie des sciences, 2009.

[16] F. Delarue, *Krylov and Safonov estimates for degenerate quasilinear elliptic PDEs,* Preprint.

[17] C. Imbert, *Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate fullynon-linear elliptic equations* Preprint.

[18] H. Ishii, Y. Yoshimura, *Demi-eigen values for uniformly elliptic Isaacs op erators* , preprint.

[19] D.Gilbarg, N.S.Trudinger Elliptic Partial Differntial equations of second order, Springer , second edition, 1983.

[20] N.V. Krylov, M. V. Safonov, *An estimate for the probability of a diffusion process hitting a set of positive measure.* (Russian) Dokl. Akad. Nauk SSSR 245 (1979), no. 1, 18–20.

[21] N.V. Krylov, M. V. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients* Izv. Akad. Nauk SSSR Ser. Mat., 1980, **44**, Issue 1, Pages 161-175

[22] P.-L. Lions, *Bifurcation and optimal stochastic control*, Nonlinear Anal. **7** (1983), no. 2, 177-207.

[23] A. Quaas, B. Sirakov, *Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators.* Adv. Math. 218 (2008), no. 1, 105-135.

[24] A. Quaas, B. Sirakov, *On the principal eigenvalues and the Dirichlet problem for fully nonlinear operators.* C. R. Math. Acad. Sci. Paris **342** (2006), no. 2, 115-118.

[25] J. Serrin,*On the Harnack inequality for linear elliptic equations.* J. Analyse Math. 4 (1955/56), 292-308.

[26] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Mathematica, vol. 111, no. 1, pp. 247-302, (1964).

[27] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations,* Communications on Pure and Applied Mathematics, vol. 20, pp. 721-747, (1967).