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Inertial Manifolds for Partial Differential Evolution Equations under Time-Discretization: Existence, Convergence, and Applications

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Dans ce travail nous étudions l'existence et la convergence de variétés inertielles pour la discrétisation en temps d'équations d'évolution. Nous montrons que pour des pas de temps petits, et sous une condition qui assure l'existence de variétés inertielles exactes (condition d'écart spectral), le problème discrétisé possède une variété inertielle de même dimension. Nous montrons la convergence de la variété approchée vers la variété exacte dans un sens fort et avec estimation d'erreur. Nos applications comprennent des équations non dissipatives, elles ne sont pas limitées au cas purement parabolique. C'est ainsi que nous considérons des équations d'amplitude du type Ginzburg–Landau et des perturbations dissipatives des équations de Korteweg–de Vries. © 1991 Academic Press, Inc.

This work is devoted to the question of existence and convergence of inertial manifolds for evolution equations under time discretization. We show that provided the time step is sufficiently small and under the condition of existence of exact inertial manifolds (the spectral gap condition), the discretized problem do have an inertial manifold with the same dimension. We show the convergence of the approximated manifolds towards the exact one in a strong sense and we give an error estimate. Our applications include nondissipative equations, they are not limited to purely parabolic equations. We consider complex amplitude equations of the type of Ginzburg–Landau equation and also dissipative perturbations of Korteweg–de Vries equations. © 1991 Academic Press, Inc.

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0. INTRODUCTION

Studies of the long time behavior of solutions to nonlinear partial differential equations have recently increased. Although this kind of problem arises from many branches of physics and mathematical physics, it is certainly the problem of turbulence that has mainly motivated these studies. Partial differential equations (PDE's) model systems with infinite number of degrees of freedom, i.e., systems whose complete description at a given time demands an infinite number of parameters. Hence evolutionary PDE's produce infinite dimensional dynamical systems.

A fundamental question then is whether or not the permanent flow (i.e., after a transient period) actually depends on a finite number of degrees of freedom. There are of course many ways to try to answer this question from a mathematical point of view. And it turns out that this research has led to introducing new concepts such as: attractors, global lyapunov exponents, determining modes, ..., and very recently that of inertial manifolds [11, 9, 15, 2] see the book by Temam [21] for a recent review on this subject.

Let us write the evolutionary PDE under consideration as follows

$$\frac{du}{dt} = N(u), \tag{0.1}$$

where N denotes an unbounded and nonlinear operator on the infinite dimensional Banach space H where $u = u(t)$ takes its values. The global attractor $\mathscr{A}$, when it exists, is the unique compact subset of H which is invariant under the evolution (0.1) and attracts all the solutions. As is known, even for finite dimensional equations, i.e., ODE's, $\mathscr{A}$ can be very complicated, and could have a complex topological structure. However, proving that $\mathscr{A}$ has finite dimension, for example finite Hausdorff dimension, and obtaining an estimate on this dimension in terms of the physical parameters, is a way to derive a bound on the actual number of degrees of freedom which is necessary to describe the permanent flows.

Unfortunately attractors have two disadvantages. First, the speed of convergence of trajectories towards an attractor can be very small, allowing complex transients and "simple" attractors. Second, in general attractors are not stable with respect to perturbations (see below). Also, from the computational point of view, since these sets can be very complicated, the problem of approximating them is difficult.

Inertial manifolds do not have these disadvantages. Indeed, these sets which are finite dimensional Lipschitz manifold invariant by the flow (0.1), attact *exponentially* the trajectories. They are stable to perturbations, as results in particular from this work. More precisely a splitting $H = H_1 \oplus H_2$

being given, where H_1 is finite dimensional, an inertial manifold $M = M_\phi$ is searched as the graph of a function $\phi: H_1 \to H_2$. Denoting by P_1 the projection on the first factor, if M is invariant by (0.1) we must have $u = p + \phi(p)$, where p is solution to

$$\frac{dp}{dt} = P_1 N(p + \phi(p)) \tag{0.2}$$

when the initial condition is taken on M. Since the trajectories of (0.1) converge rapidly to M, it follows that the PDE (0.1) reduces to an ODE and the dimension of H_1 is a bound on the actual number of degrees of freedom.

Investigations on the qualitative behavior of solutions to nonlinear evolution P.D.E.'s are now mainly computational. For that purpose, systems of the type (0.1) are time-discretized, and replaced by iterations (i.e., discrete dynamical systems), for example,

$$u^{n+1} = u^n + \tau N_\tau(u^n), \tag{0.3}$$

where $\tau > 0$ represents here the time step. Under general hypotheses (mainly that (0.1) is dissipative, we shall define this latter) (0.1) admits a global attractor $\mathscr{A}$, and provided τ is sufficiently small, (0.3) admits also a global attractor $\mathscr{A}_\tau$. As a result [14, 21] the correspondance $\tau \to \mathscr{A}_\tau$ is only upper semicontinuous in 0, i.e., given a neighborhood V of $\mathscr{A}$, $\mathscr{A}_\tau \subset V$ for sufficiently small τ; the converse being false is general. It is what we meant when we said that attractors are not stable to perturbations.

An inertial manifold $M_\tau = M_{\phi_\tau}$ for (0.3) is then the graph of a Lipschitz function ϕ_τ from H_1 into H_2, invariant by (0.3), i.e., if $u^0 \in M_\tau$ then $u^n \in M_\tau$, $\forall n \geqslant 0$, and which attracts exponentially the solutions of (0.3). Taking $u^0 = P_1 u^0 + \phi_\tau(P_\tau u^0) \in M_\tau$, we have $p^0 = P_1 u^0$ and

$$p^{n+1} = p^n + \tau P_1 N_\tau(p^n + \phi_\tau(p^n)). \tag{0.4}$$

Now (0.4) is a discrete, finite dimensional dynamical system, fitted for numerical investigations. Its long time behavior, i.e., as $n \to +\infty$, will represent that of (0.1) if the inertial manifolds are close.

The question of approwimating (0.1) by (0.3) on *large time intervals* is not an easy task, indeed, recall that classical error estimates (even for ODE's) are of the form

$$\|u(n\tau) - u^n\| \leqslant C\tau^r, \qquad 0 \leqslant n \leqslant N,$$

where r is the order of the method, but the constant C grows exponentially with N and therefore as $n \to +\infty$ this estimate vanishes. Moreover, in systems of interests, (0.1) presents sensitivity to initial conditions and then it is expected that indeed $u(n\tau)-u^n$ is not small for large n, as small as τ is (see Beyn [1] for positive results in the neighborhood of an hyperbolic fixed point, in the ODE case).

Before describing the results of this work which pertain to these questions, we recall that (0.1) is said to be dissipative if the following occurs. Let $S(t)$ denote the semi group of solutions to (0.1), i.e., $u(t)=S(t)\,u(0)$, $\forall t\geqslant 0$. A bounded set B_a in H is said to be absorbing for $\{S(t)\}_{t\geqslant 0}$ in H if for every set B bounded in H, there exists $T(B)\geqslant 0$ such that $S(t)B\subset B_a$, $\forall t\geqslant T(B)$. Then (0.1) is dissipative if $S(t)$ possesses a bounded absorbing set.

The main results of this work are as follows. We show for the class of evolution equations (0.1) that we consider, that suitable discretisations (0.3) possess inertial manifolds. The dimension of this manifold M_τ, i.e., the splitting of H, can be taken the same as for that of the continuous equation M. Furthermore we show that M_τ and M are close in a strong sense, i.e., if we denote by $M=M_\phi$ and $M_\tau=M_{\phi_\tau}$, we give an error estimate on the norm of the difference $\phi-\phi_\tau$. Finally two classes of applications are given. These are the following: a class of complex amplitude equations (Ginzburg–Landau like PDE's) and equations related to the Korteweg–de Vries–Burgers equations. We note that our setting is not limited to dissipative equations, as it was in previous works on this subject. Moreover we can sometimes consider the original PDE without using modifications of the nonlinear terms in (0.1).

The main results of this work have been announced in a Note aux Comptes Rendus [22].

1. The Continuous Case

In this section we give the precise hypotheses on the abstract framework we shall use. Also we briefly set the problem and give in Theorem 1.1 a condition (similar to that in [11, 9, 21]) which ensures the existence of an inertial manifold for (0.1). The principle of the proof of Theorem 1.1 is rapidly pictured, mainly in order to introduce some notations we shall need in the remainder of the paper.

1.1. *Notations and Setting of the Problem*

We are given on an infinite dimensional real separable Hilbert space H, with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$, a linear closed unbounded positive self-adjoint operator A in H with domain $D(A)\subset H$. We assume that $v\to|Av|$ is a norm on $D(A)$ equivalent to the graph-norm and that A is an isomorphism from $D(A)$ onto H, A^{-1} being a *compact* operator on H.

Hence there exists a complete orthonormal family $\{w_j\}_{j=1}^{\infty}$ in H made of eigenfunctions of A,

$$\begin{aligned} &Aw_j = \lambda_j w_j, \qquad j=1, ... \\ &0<\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_j \to \infty \qquad \text{as} \quad j \to \infty, \end{aligned} \tag{1.1}$$

where the λ_j's are the associated eigenvalues repeated according to their multiplicity. We denote by $\sigma(A) = \{\Lambda_k\}_{k=1}^{\infty}$, $\Lambda_1 < \Lambda_2 < \cdots$ the set of distinct eigenvalues, Λ_k being of finite multiplicity m_k. The spectral projectors R_Λ and P_Λ are defined as usual

$$R_\Lambda v = \sum_{j:\, \lambda_j = \Lambda} (v, w_j)\, w_j, \; P_\Lambda = \sum_{\lambda \leqslant \Lambda} R_\lambda. \tag{1.2}$$

Recall that, given $f\colon \mathbb{C} \to \mathbb{C}$, the operator $f(A)$ is defined as

$$f(A) = \sum_\Lambda f(\Lambda)\, R_\Lambda v \qquad \text{with} \quad D(f(A)) = \left\{ v, \sum_\Lambda |f(\Lambda)|^2\, |R_\Lambda v|^2 < \infty \right\}.$$

We are also given C, a linear bounded and squew-symmetric operator from $D(A^{s_0})$, $s_0 \in \mathbb{R}_+^*$ into H. We assume that $C \in \mathscr{L}(D(A^{\alpha + s_0}), D(A^\alpha))$ for every $\alpha \in \mathbb{R}$ and C and A commutes:

$$AC = CA. \tag{1.3}$$

It follows that $C_\Lambda = R_\Lambda C$ maps $R_\Lambda H$ into itself and since this space is finite dimensional, C_Λ is simply a squew-symmetric matrix. It is therefore also possible to define $f(C)$, for f as before. We note that sometimes we need to consider the complexification $H_{\mathbb{C}}$ of H, but we shall denote by the same symbols an object and its complexification. This should not produce any confusion.

We consider the following differential equation

$$\frac{du}{dt} + Au + Cu + F(u) = 0, \tag{1.4}$$

$$u(0) = u_0, \tag{1.5}$$

where F is a Lipschitz function from $D(A^\alpha)$ into $D(A^{\alpha-\gamma})$ for some given $\alpha \in \mathbb{R}$ and $\gamma \in [0, \frac{1}{2}]$,

$$|A^{-\gamma}(F(v) - F(w))|_\alpha \leqslant L_F\, |v - w|_\alpha, \qquad \forall v, w \in D(A^\alpha), \tag{1.6}$$

where here and in the sequel we denote by

$$(v, w)_\alpha = (A^\alpha v, A^\alpha w), \; |v|_\alpha = |A^\alpha v|$$

the scalar product and norm on $D(A^\alpha)$. For convenience in the notations we shall assume that L_F in (1.6) has been chosen in order that we also have

$$|A^{-\gamma}F(v)|_\alpha \leqslant L_F(1+|v|_\alpha), \qquad \forall v \in D(A^\alpha). \tag{1.7}$$

Following [11], an *inertial manifold* for (1.4) is a finite dimensional Lipschitz manifold $M \subset D(A^\alpha)$ which enjoys the following properties. This set is positvely invariant by (1.4) and attracts exponentially all the solutions, i.e.,

$$u_0 \in M \Rightarrow u(t) \in M, \qquad \forall t \geqslant 0, \tag{1.8}$$

$$\forall R>0, \exists \sigma>0, C\geqslant 0 \text{ such that } \forall t \geqslant 0, u_0 \in D(A^\alpha)$$
$$\text{with } |u_0|_\alpha \leqslant R,\ d_\alpha(u(t), M) = \operatorname*{Inf}_{m \in M} |u(t)-m|_\alpha \leqslant Ce^{-\sigma t}. \tag{1.9}$$

The hypothesis (1.6) seems very restrictive, at least at first sight. Besides the fact that for some applications this holds true, actually one proceeds in a similar fashion as for the study of invariant manifolds in the neighborhood of a fixed point and uses the truncation method to derive (1.6) from the original equation. Loosely speaking, considering a nonlinear dissipative equation that admits a bounded absorbing set B_a, the genuinely nonlinear terms are modified outside of B_a in order that (1.6) holds true. Hence, since the long time behavior of the original equation is the question we aim to consider, this procedure leads to a relevant problem (see the applications for the precise construction).

Due in particular to (1.6), it is well known that for $u_0 \in D(A^\alpha)$, the Cauchy problem (1.4)–(1.5) admits a unique solution with

$$u \in \mathscr{C}([0, +\infty[; D(A^\alpha)) \cap L^2(0, T; D(A^{\alpha+1/2})), \qquad \forall T>0. \tag{1.10}$$

Moreover if $u_0 \in D(A^{\alpha+1/2})$, this solution is more regular:

$$u \in \mathscr{C}([0, \infty[; D(A^{\alpha+1/2})) \cap L^2(0, T; D(A^{\alpha+1})), \qquad \forall T>0. \tag{1.11}$$

We denote by $S(t)$ the time t map on $D(A^\alpha)$ or $D(A^{\alpha+1/2})$,

$$S(t)u_0 = u(t), \qquad t \geqslant 0. \tag{1.12}$$

These mappings are Lipschitz continuous and bounded on $D(A^\alpha)$ and $D(A^{\alpha+1/2})$ and since (1.4) is automnomous the family $\{S(t), t\geqslant 0\}$ forms a semigroup,

$$S(0)=I,\ S(t_1+t_2)=S(t_1)\,S(t_2). \tag{1.13}$$

1.2. *Construction of an Inertial Manifold*

For $\Lambda \in \sigma(A)$, we set for the sake of simplicity in the notations, $P = P_\Lambda$, $Q = I - P_\Lambda$. The inertial manifold M is searched as the graph of a lipschizian function from PH into $QD(A^\alpha)$,

$$M = M_\phi = \{ p + \phi(p), p \in PH \}. \tag{1.14}$$

Besides the fact that all the methods presently known for constructing such manifolds lead to this type of set, this form is suitable from a practical point of view since by (1.8) the semi flow $S(t)$ reduces on M to a flow. Namely the flow of the ODE

$$\frac{dp}{dt} + Ap + Cp + PF(p + \phi(p)) = 0. \tag{1.15}$$

Indeed, thanks to (1.3), P and C commutes and

$$S(t)\, M_\phi \subset M_\phi, \qquad \forall t \geqslant 0 \tag{1.16}$$

shows that $p = p(t) = Pu(t)$ must satisfy (1.15) for nonnegative t. But now (1.15) is a finite dimensional ODE on PH and since ϕ is Lipschitz it follows from (1.6) that (1.15) possesses global solutions which are defined on the whole real line. This induces a group on PH,

$$S_\phi(t)\, p_0 = p(t), \qquad p(0) = p_0; \qquad \forall t \in \mathbb{R}. \tag{1.17}$$

Therefore we have

$$S(t) M_\phi = M_\phi, \qquad \forall t \in \mathbb{R}, \tag{1.18}$$

where

$$S(t)\{m, \phi(m)\} = \{S_\phi(t)m, \phi(S_\phi(t)m)\}, \qquad \forall m \in M_\phi, \forall \in \mathbb{R}. \tag{1.19}$$

Let us now state a result of existence of such invariant manifolds. This theorem slightly generalizes previous results [11, 9, 21] in various directions: we do not assume that either F is uniformly bounded or $F(u)$ vanishes for large $|u|_\alpha$; we are able to consider the case $C \neq 0$. We also include a C^1 smoothness result.

THEOREM 1.1. *If N satisfies*

$$\begin{aligned} \Lambda_{N+1} &\geqslant 3L_F^2 \Lambda_1^{2\gamma - 1}/2, \\ \Lambda_{N+1} - \Lambda_N &\geqslant 30 L_F (\Lambda_N^\gamma + \Lambda_{N+1}^\gamma), \end{aligned} \tag{1.20}$$

then there exists $\phi \in \mathscr{C}(P_{\Lambda_N}H, (I-P_{\Lambda_N})\, D(A^\alpha))$ *such that*

$$|\phi(p_1)-\phi(p_2)|_\alpha \leqslant |p_1-p_2|_\alpha/4, \qquad \forall p_1, p_2 \in P_{\Lambda_N}H \tag{1.21}$$

and M_ϕ *is an inertial manifold for* (1.4). *Furthermore if* F *is a* C^1 *function from* $D(A^\alpha)$ *into* $D(A^{\alpha-\gamma})$, *then* ϕ *is* C^1 *from* $P_{\Lambda_N}H$ *into* $(I-P_{\Lambda_N})\, D(A^\alpha)$.

Remark 1.1. The dimension of M_ϕ is $\sum_{j=1}^N m_j$.

We briefly give the main steps of the proof of this result, refering to [5] for a complete proof. The main goal of what follows is to introduce some notations and complementary properties which we shall need in the remainder of this article.

Given $l \in [0, \infty[$, we denote by $\mathscr{F}_l$ the following set of functions from PH into $QD(A^\alpha)$,

$$\mathscr{F}_l = \{\phi,\ \|\phi\|_\alpha \leqslant l,\ \mathrm{lip}_\alpha(\phi) \leqslant l\}, \tag{1.22}$$

where

$$\|\phi\|_\alpha = \mathrm{Sup}\{|\phi(p)|_\alpha/(1+|p|_\alpha),\ p \in PH\}, \tag{1.23}$$

and

$$\mathrm{Lip}_\alpha(\phi) = \mathrm{Sup}\{|\phi p_1)-\phi(p_2)|_\alpha/|p_1-p_2|_\alpha,\ p_i \in PH\}. \tag{1.24}$$

For ϕ given in $\mathscr{F}_l$, we can solve (1.15) and construct by (1.17) the mappings $S_\phi(t)$ on PH.

We set

$$(\mathscr{T}\phi)(p_0) = -\int_{-\infty}^0 e^{(A+C)\sigma} QF(S_\phi(\sigma)\, p_0 + \phi(S_\phi(\sigma)\, p_0))\, d\sigma, \tag{1.25}$$

and this defines a map from $\mathscr{F}_l$ into $\mathscr{C}(PH, QD(A^\alpha))$ as soon as $\Lambda_{N+1} > \Lambda_N + \Lambda_N^\gamma L_F(1+l)$. It is easy to check that if M_ϕ satisfies (1.18), it is necessary that $\mathscr{T}\phi = \phi$. And, indeed, ϕ is found as a fixed point of $\mathscr{T}$ in $\mathscr{F}_{1/4}$: the main consequence of (1.20) is that $\mathscr{T}$ maps $\mathscr{F}_{1/4}$ into itself and is strictly contracting on this set. This allows us to construct M_ϕ. The exponential attraction property (1.9) is not a byproduct. It follows from a property that enjoys (1.4) under (1.20): the cone property. This property was introduced in [9]. It can be phrased as follows. We denote by $\mathscr{C}_\kappa$ the cone in $D(A^\alpha)$,

$$\mathscr{C}_\kappa = \{v \in D(A^\alpha),\ |Pv|_\alpha \geqslant \kappa\ |Qv|_\alpha\}. \tag{1.26}$$

We say with [9], that (1.4) enjoys the $\mathscr{C}_\kappa$-cone property if $\mathscr{C}_\kappa$ is stable by $S(t)$ (see (1.27) below) and considering two trajectories $u_1(t)$ and $u_2(t)$ of

(1.4) only two situations can occur: either $u_1(t)-u_2(t)$ enters in a finite time in $\mathscr{C}_\kappa$ or $A^\alpha(u_1-u_2)$ tends to zero exponentially as $t \to +\infty$. The stability property quoted above is

$$u_1(0)-u_2(0) \in \mathscr{C}_\kappa \Rightarrow u_1(t)-u_2(t) \in \mathscr{C}_\kappa, \qquad \forall t \geqslant 0. \tag{1.27}$$

Now under condition (1.20), it turns out that (1.4) has the $\mathscr{C}_4$-cone property and there exist two constants K and $\sigma > 0$ such that

$$d_\alpha(S(t)\,u_0, M_\phi) \leqslant K d_\alpha(u_0, M_\phi)\, e^{-\sigma t}, \qquad \forall u_0 \in D(A^\alpha), \forall t \geqslant 0. \tag{1.28}$$

This property shows (1.9). Actually (1.28) is stronger.

Finally the smoothness result on ϕ, i.e., that ϕ is in fact a C^1-function, is obtained by using a fiber contraction theorem (see, e.g., [19]). We refer to [5] for the details.

2. Construction of an Inertial Manifold in the Discrete Case

In this section we replace the continuous Eqs. (1.4)–(1.5) by numerical schemes. This produces discrete dynamical systems on an infinite dimensional space. Under the condition of existence for an inertial manifold in the continuous case stated in Theorem 1.1, we construct inertial manifolds for the discrete equations.

2.1. *The Self-Adjoint Case*

In this section we consider the case where $C=0$. Apart from the fact that it is a relevant case in view of the applications, the results we prove here are of interest for the general case we shall address in Section 2.2.

2.1a. *A Semi-Implicit One Step Method.* Given $u^0 \in D(A^\alpha)$ and $\tau > 0$, we introduce the sequence $\{u^n, n \in \mathbb{N}\} \subset D(A^\alpha)$ by the formula

$$(u^{n+1}-u^n)/\tau + Au^{n+1} + F(u^n) = 0, \qquad n \geqslant 0. \tag{2.1}$$

Since the operator $(I+\tau A)$ is an isomorphism from $D(A^{\alpha+1})$ onto $D(A^\alpha)$, we can set

$$R(\tau) = (I+\tau A)^{-1} \tag{2.2}$$

and rewrite (2.1) as

$$u^{n+1} = R(\tau)(u^n - \tau F(u^n)), \qquad \forall n \geqslant 0. \tag{2.3}$$

Our aim in this section is to find a function ϕ_τ mapping PH into $QD(A^\alpha)$ and such that $M_\tau = M_{\phi_\tau}$ is an inertial manifold for (2.1). For the moment,

$P = P_N$, where N is arbitrary. But as a result, we shall prove that, provided N satisfies the hypothesis of Theorem 1.1, we are able to consider the same splitting of $D(A^\alpha)$ as in the continuous case: $P_N D(A^\alpha) \oplus Q_N D(A^\alpha)$.

The mapping S^τ on $D(A^\alpha)$

$$S^\tau v = R(\tau)(v - \tau F(v)), \qquad \forall v \in D(A^\alpha) \tag{2.4}$$

is Lipschitz-continuous thanks to (1.6). By mimicking the continuous case we set the following definition.

DEFINITION 2.1. With the previous notations, an inertial manifold for (2.1) is a finite dimensional Lipschitz manifold $M \subset D(A^\alpha)$ which satisfies

$$S^\tau M \subset M, \tag{2.5}$$

$\forall R > 0, \exists \sigma > 0, C \geqslant 0$ such that

$$d_\alpha((S^\tau)^n u^0, M) = d_\alpha(u^n, M) \leqslant Ce^{-\sigma n\tau}, \forall u \geqslant 0, \forall u \geqslant 0, |u|_\alpha \leqslant R. \tag{2.6}$$

In the discrete case, too, M will be searched as the graph of a Lipschitz function ϕ and therefore the infinite dimensional recursion formula (2.1) will be replaced on $M = M_\phi$ by the *finite dimensional iteration*

$$(p^{n+1} - p^n)/\tau + Ap^{n+1} + PF(p^n + \phi(p^n)) = 0. \tag{2.7}$$

This can also be written as

$$p^{n+1} = S_\phi^\tau p_n, \qquad n \geqslant 0, \tag{2.8}$$

where S_ϕ^τ is the Lipschitz continuous map on PH defined by

$$S_\phi^\tau p = R(\tau)(p - \tau PF(p + \phi(p))). \tag{2.9}$$

Taking $\phi \in \mathscr{F}_l$, i.e., assuming that ϕ is l-Lipschitzian, it is clear on (2.9) that S_ϕ^τ can be inverted for small τ. More precisely, one easily verifies the following result:

LEMMA 2.1. *For $\phi \in \mathscr{F}_l$ and provided*

$$0 < \tau < L_F^{-1}(1 + l)^{-1} \Lambda_N^{-\gamma}, \tag{2.10}$$

the mapping S_ϕ^τ is a homeomorphism of PH, $P = P_N$. Moreover S_ϕ^τ and $(S_\phi^\tau)^{-1}$ are Lipschitzian on PH.

When (2.10) holds, we deduce from this lemma that for $M = M_\phi$ (2.5) is equivalent to

$$(S^\tau)^n M = M, \qquad \forall n \in \mathbb{Z}, \tag{2.11}$$

and (compare with (1.19))

$$(S_\phi^\tau)^n(m+\phi(m)) = (S_\phi^\tau)^n m + \phi((S_\phi^\tau)^n m), \qquad \forall n \in \mathbb{Z}, \forall m \in M_\phi. \quad (2.12)$$

2.1b. *Existence of an Inertial Manifold.* In this section our first aim is to derive the necessary form of $\phi \in \mathscr{F}_l$ such that (2.11) holds true for $M = M_\phi$. For we take $u^0 = p^0 + \phi(p^0) \in M$ and project (2.3) on PH and QH. Setting $p^n = (S_\phi^\tau)^n p^0$, $q^n = \phi(p^n)$, we find

$$p^{n+1} = R(\tau)(p^n - \tau PF(p^n + \phi(p^n))), \qquad n \in \mathbb{Z}, \quad (2.13)$$

$$q^{n+1} = R(\tau)(q^n - \tau QF(p^n + \phi(p^n))), \qquad n \in \mathbb{Z}. \quad (2.14)$$

From this last relation it follows that for $m \leqslant n$,

$$q^n = R(\tau)^{n-m} q^m - \tau \sum_{k=m+1}^{n} R(\tau)^{n+1-k} QF(p^{k-1} + \phi(p^{k-1})). \quad (2.15)$$

We are going to see that provided $\Lambda - \lambda$ is sufficiently large and τ is sufficiently small (see (2.23)–(2.24) below), the first term in the right hand side of (2.15) converges towards 0 as $m \to -\infty$ leading to

$$q^n = -\tau \sum_{k=-\infty}^{n} R(\tau)^{n+1-k} QF(p^{k-1} + \phi(p^{k-1})), \qquad \forall n \in \mathbb{Z}. \quad (2.16)$$

Taking (2.16) for granted, we see that since $q^0 = \phi(p^0)$ we have

$$\phi(p^0) = -\tau \sum_{k=1}^{+\infty} R(\tau)^k QF((S_\phi^\tau)^{-k} p^0 + \phi(S_\phi^\tau)^{-k} p^0). \quad (2.17)$$

Introducing the mapping $\mathscr{T}_\tau$ on $\mathscr{F}_l$, τ satisfying (2.10),

$$(\mathscr{T}_l \phi)(p^0) = -\tau \sum_{k=1}^{+\infty} R(\tau)^k QF((S_\phi^\tau)^{-k} p^0 + \phi((S_\phi^\tau)^{-k} p^0)), \quad (2.18)$$

we see that (2.17) reads $\phi = \mathscr{T}_\tau \phi$ and show that ϕ can be searched as a fixed point for $\mathscr{T}_\tau$. We note that this formula is the analog of (1.24). Indeed the convergence results proved in the third section are based on the fact that $\mathscr{T}_\tau$ converges towards $\mathscr{T}$ as $\tau \to 0$ in a suitable sense, see, for example, (3.20).

The main result of Section 2.1, which provides existence of an inertial manifold for the scheme (2.1) is as follows.

THEOREM 2.1. *We assume that $N \geqslant 1$ is such that*

$$\Lambda_{N+1} \geqslant 3L_F^2 \Lambda_1^{2\gamma-1}/2,$$

$$\Lambda_{N+1} - \Lambda_N \geqslant 30 L_F(\Lambda_N^\gamma + \Lambda_{N+1}^\gamma).$$

For every $\tau > 0$, such that $\tau \Lambda_{N+1} \leqslant 1$, the discrete infinite dimensional dynamical system on $D(A^{\alpha})$,

$$(u^{n+1} - u^n)/\tau + Au^{n+1} + F(u^n) = 0, \qquad n \geqslant 0,$$

possesses an inertial manifold M_τ which is the graph of a Lipschitz mapping from $P_N H$ into $Q_N D(A^{\alpha})$. Furthermore, there exists two constants c_0 and $\sigma > 0$ such that

$$d_\alpha(u^n, M_\tau) \leqslant c_0 e^{-\sigma n}\, d_\alpha(u^0, M_\tau), \qquad \forall u^0 \in D(A^{\alpha}).$$

The proof of this result is given in Section 2.1e. In the next section, we begin by giving a sufficient condition that allows us to find a fixed point for the mapping $\mathscr{T}_\tau$ on $\mathscr{F}_l$ (Proposition 2.1). This produces a natural candidate for the manifold M_τ in Theorem 2.1. Then Section 2.1d is devoted to the study of the norm of the difference of two solutions to (2.1) (Proposition 2.2), having in mind to show the exponential attraction property of M_τ. These two Propositions, besides their own interest, are the main ingredients in the proof of Theorem 2.1.

2.1c. *Construction of an Invariant Manifold.* We are going to show that for sufficiently small τ, the mapping $\mathscr{T}_\tau$ is a contraction on $\mathscr{F}_{1/4}$ and this provides the existence of an invariant manifold $M_\tau = M_{\phi_\tau}$, where $\phi_\tau = \mathscr{T}_\tau \phi_\tau$.

PROPOSITION 2.1. *We assume that N satisfies*

$$\Lambda_{N+1} - \Lambda_N \geqslant 6L_F(\Lambda_N^{\gamma}(1+l) + \Lambda_{N+1}^{\gamma}(1+l^{-1})). \tag{2.19}$$

Then for every $\tau > 0$ such that

$$\tau \Lambda_{N+1} \leqslant 1, \tag{2.20}$$

the mapping $\mathscr{T}_\tau$ maps $\mathscr{F}_l$ into itself and

$$\|\mathscr{T}_\tau \phi_1 - \mathscr{T}_\tau \phi_2\|_\alpha \leqslant \kappa\, \|\phi_1 - \phi_2\|_\alpha, \qquad \forall \phi_i \in \mathscr{F}_l, \tag{2.21}$$

with $\kappa = l(2 + (1+l)^{-1})$.

Taking $l = \frac{1}{4}$, we deduce the following

COROLLARY 2.1. *We assume that N satisfies*

$$\Lambda_{N+1} - \Lambda_N \geqslant 10L_F(\Lambda_N^{\gamma} + \Lambda_{N+1}^{\gamma}).$$

Then for every $\tau > 0$ satisfying (2.20), the mapping $\mathscr{T}_\tau$ is a strict contraction from $\mathscr{F}_{1/4}$ into itself.

Before giving the proof of Proposition 2.1, we derive an estimate on p^n satisfying (2.13):

LEMMA 2.2. *We assume that* (2.10) *holds true and set*

$$\eta = (1 + \tau\Lambda_N)(1 - \tau\Lambda_N^{\gamma} L_F(1 + l))^{-1},$$
$$\beta = \tau^{\gamma} L_F(1 + l)(1 - \tau^{\gamma} LF(1 + l))^{-1}.$$

The solutions of (2.13) *satisfy*

$$|p^m|_\alpha \leqslant \eta^{-m} |p^0|_\alpha + \beta(\eta^{-m} - 1)/(\eta - 1), \qquad \forall m \leqslant 0. \tag{2.22}$$

We set here and in the sequel, $\lambda = \Lambda_N$, $\Lambda = \Lambda_{N+1}$, $P = P_N$, $Q = I - P_N = Q_N$. We rewrite (2.13) as

$$R(\tau)^{-1} p^{n+1} = p^n - \tau PF(p^n + \phi(p^n)).$$

The scalar product of this identity in $D(A^\alpha)$ with p_n leads to

$$|p^n|_\alpha^2 \leqslant \tau(PF(p^n + \phi(p^n)), p^n)_\alpha + (1 + \tau\lambda) |p^n|_\alpha |p^{n+1}|_\alpha. \tag{2.23}$$

Then using (1.6) and (1.7) we find, since $\phi \in \mathscr{F}_l$, that

$$|A^{-\gamma} F(p^n + \phi(p^n)|_\alpha \leqslant L_F(1 + l)(1 + |p^n|_\alpha).$$

Hence

$$|(PF(p^n + \phi(p^n), p^n)_\alpha| \leqslant \lambda^\gamma L_F(1 + l)(1 + |p^n|_\alpha) |p^n|_\alpha,$$

which produces with (2.23),

$$|p^n|_\alpha \leqslant \eta |p^{n+1}|_\alpha + \beta,$$

where η and β are given in Lemma 2.2. Then the formula (2.22) follows readily by induction on $m \leqslant 0$.

In deriving the necessary form of an inertial manifold, we have assumed in (2.15) that $R(\tau)^{n-m} q^m$ goes to 0 as $m \to -\infty$. It is a consequence of (2.19) and (2.22). Indeed, by (2.22) and the fact that $\phi \in \mathscr{F}_l$, there exists a constant C such that $|q^m|_\alpha = \phi(p^m)| \leqslant C\eta^{-m}$. On the other hand $|R(\tau)^{n-m} q|_\alpha \leqslant (1 + \tau\Lambda)^{m-n} |q|_\alpha$ for $q \in QD(A^\alpha)$; therefore

$$|R^{n-m} q^m|_\alpha \leqslant C(1 + \tau\Lambda)^n ((1 + \tau\Lambda)/\eta)^m. \tag{2.23$'$}$$

Finally we note that

$$\tau\Lambda \leqslant 1, \Lambda - \lambda > 2\lambda^\gamma L_F(1 + l) \Rightarrow 1 + \tau\Lambda > \eta. \tag{2.24}$$

And this proves our claim.

We are now in a position to prove Proposition 2.1. We take $\phi \in \mathscr{F}_l$ and assume that (2.19) holds true:

$$\Lambda - \lambda \geqslant 6L_F((1+l)\lambda^\gamma + (1+l^{-1})\Lambda^\gamma). \tag{2.25}$$

It follows that $\Lambda > 2(1+l)\, L_F \lambda^\gamma$ and thanks to (2.20), (2.10) holds true. This allows us to define $\mathscr{T}_\tau$ by the formula 2.18) since (2.24) is now satisfied. It is in fact convenient to rewrite this formula as

$$(\mathscr{T}_\tau \phi)(p^0) = -\tau \sum_{k=1}^{\infty} R(\tau)^k Q G_\phi(p^{-k}), \tag{2.26}$$

where $p^{-k} = (S^\tau_\phi)^{-k} p^0$ and

$$G_\phi(p) \equiv F(p + \phi(p)). \tag{2.27}$$

In order to prove Proposition 2.1, we have to show that (i) $\mathscr{T}_\tau$ maps $\mathscr{F}_l$ into itself and that (ii) $\mathscr{T}_\tau$ verifies (2.21). This leads us to consider the expressions $\mathscr{T}_\tau \phi^1 - \mathscr{T}_\tau \phi^2$ and $(\mathscr{T}_\tau \phi)\, p_1^0 - (\mathscr{T}_\tau \phi)\, p_2^0$. With regards to (2.26), we have to estimate

$$(\delta p)_k \equiv (S^\tau_{\phi_1})^k p_1^0 - (S^\tau_{\phi_2})^k p_2^0, \qquad k \leqslant 0. \tag{2.28}$$

This is done in the following lemma whose proof is postponed.

LEMMA 2.3. *With the hypotheses and notations in Lemma 2.2, $(\delta p)_n$ in (2.28) is estimated as*

$$\begin{aligned} |(\delta p)_n|_\alpha \leqslant \eta^{-n}\, &|p_1^0 - p_2^0|_\alpha \\ &- n\eta^{-n} pp(1+l)^{-1}\, \|\phi_1 - \phi_2\|_\alpha (1 + |p_1^0|_\alpha), \qquad \forall n \leqslant 0. \end{aligned} \tag{2.29}$$

We take $\phi \in \mathscr{F}_l$ and $p_1^0, p_2^0 \in H$. According to (2.26),

$$(\mathscr{T}_\tau \phi)(p_1^0) - (\mathscr{T}_\tau \phi)(p_2^0) = -\tau \sum_{k=1}^{\infty} R(\tau)^k Q(G_\phi(p_1^{-k}) - G_\phi(p_2^{-k})),$$

then using (1.6) and (2.29) we find

$$\begin{aligned} &|(\mathscr{T}_\tau \phi)(p_1^0) - (\mathscr{T}_\tau \phi)(p_2^0)| \\ &\qquad \leqslant \tau L_F (1+l)|\, p_1^0 - p_2^0| \sum_{k=1}^{\infty} \eta^k\, |A^\gamma R(\tau)^k Q|_{\mathscr{L}(D(A^\alpha))}. \end{aligned} \tag{2.30}$$

The norm in $\mathscr{L}(D(A^\alpha))$ of the operator $A^\gamma R(\tau)^k Q$, $k \geqslant 1$, is bounded as follows,

$$|A^\gamma R(\tau)^k Q|_{\mathscr{L}(D(A^\alpha))} \leqslant (\Lambda^\gamma + \tau^{-\gamma} k^{-\gamma})(1 + \tau \Lambda)^{-k} \tag{2.31}$$

which follows in turn from the inequality

$$\xi^{\gamma}(1+\tau\xi)^{-k} \leqslant (\Lambda^{\gamma}+\tau^{-\gamma}k^{-\gamma})(1+\tau\Lambda)^{-k}$$

valid for $\xi \geqslant \Lambda$, $k \geqslant 1$, $\gamma \in [0, \frac{1}{2}]$ and $\tau > 0$. Hence (2.30) and (2.31) show that

$$|\mathscr{T}_\tau\phi)(p_1^0) - (\mathscr{T}_\tau\phi)(p_2^0)|_\alpha \leqslant \omega\, |p_1^0 - p_2^0|_\alpha, \tag{2.32}$$

where

$$\omega = \tau L_F(1+l) \sum_{k=1}^{\infty} (\Lambda^{\gamma}+\tau^{-\gamma}\kappa^{-\gamma})\, \eta^k (1+\tau\Lambda)^{-k}. \tag{2.33}$$

Our aim now is to prove that

$$\{\text{Eq. (2.25) and } \tau\Lambda \leqslant 1\} \Rightarrow \omega \leqslant l. \tag{2.34}$$

We set $r = \eta(1+\tau\Lambda)^{-1}$ and claim that $r < 1$. Indeed this is equivalent to (η is given in Lemma 2.2)

$$1 + \tau\lambda < (1+\tau\Lambda)(1 - \tau L_F(1+l)\, \lambda^{\gamma})$$

which holds true thanks to (2.20) and (2.25). Returning to (2.33), we have

$$\omega \leqslant \tau L_F(1+l) \left\{ \frac{2\Lambda^{\gamma}}{1-r} + \tau^{-\gamma} \sum_{k=1}^{\infty} k^{-\gamma} r^k \right\}. \tag{2.35}$$

The infinite sum is bounded by the integral $\int_0^{+\infty} x^{-\gamma} r^x \, dx = |\mathrm{Log}\, r|^{\gamma-1} \int_0^{+\infty} x^{-\gamma} e^{-x} \, dx \leqslant 2\, |\mathrm{Log}\, r|^{\gamma-1}$ and this leads to $\omega \leqslant \tau L_F(1+l)$ $\{\Lambda^{\gamma} r/(1-r) + 2\tau^{-\gamma} |\mathrm{Log}\, r|^{\gamma-1}\}$. Then, since $|\mathrm{Log}\, r| \geqslant 1-r$, we find

$$\omega \leqslant \tau L_F(1+l) \left\{ \frac{\Lambda^{\gamma} r}{1-r} + 2\tau^{-\gamma}(1-r)^{\gamma-1} \right\},$$

and using that $1 - r \leqslant \tau\Lambda$, we conclude that $\omega \leqslant 3\tau L_F(1+l)\, \Lambda^{\gamma}(1-r)^{-1}$. Reporting the value of r, we finally have

$$\omega \leqslant \frac{3L_F(1+l)\, \Lambda^{\gamma}(1+\tau\Lambda)}{\Lambda - \lambda - \lambda^{\gamma} L_F(1+l)(1+\tau\Lambda)}, \tag{2.36}$$

and (2.34) follows readily. In view of (2.32), we have shown that $\mathrm{Lip}_\alpha(\mathscr{C}_\tau(\phi) \leqslant l$. In order to show that $\mathscr{T}_\tau\phi \in \mathscr{F}_l$, it remains to verify that

$|(\mathscr{T}_\tau\phi)(p^0)|_\alpha \leqslant l(1+|p^0|_\alpha)$, for every $p^0 \in PH$. This calculation is almost the same as before. Indeed, by (1.7) and (2.26), we have

$$|(\mathscr{T}_\tau\phi)(p^0)|_\alpha \leqslant \tau(1+l)\, L_F \sum_{k=1}^{\infty} |A^\gamma R(\tau)^\gamma Q|_{\mathscr{L}(D(A^\alpha))}(1+|p^{-k}|_\alpha). \tag{2.37}$$

Thanks to (2.22), we have

$$1+|p^{-k}|_\alpha \leqslant \eta^k(1+|p^0|_\alpha), \qquad \forall k \geqslant 0. \tag{2.38}$$

Reporting this estimate in (2.37) and using (2.33) we find $|\mathscr{T}_\tau\phi(p^0)|_\alpha \leqslant \omega(1+|p^0|_\alpha)$. Thanks to (2.32), we conclude that $\mathscr{T}_\tau\phi \in \mathscr{F}_l$.

Concerning the inequality (2.21) in Proposition 2.1, we proceed as follows. We take $\phi_1, \phi_2 \in \mathscr{F}_l$ and $p_1^0 = p_2^0 = p^0 \in PH$. According to (2.26),

$$(\mathscr{T}_\tau\phi_1)(p^0) - (\mathscr{T}_\tau\phi_2)(p^0) = -\tau \sum_{k=1}^{\infty} R(\tau)^k Q(G_{\phi_1}(p_1^{-k}) - G_{\phi_2}(p_2^{-k})). \tag{2.39}$$

We write

$$G_{\phi_1}(p_1) - G_{\phi_2}(p_2) = G_{\phi_1}(p_1) - G_{\phi_1}(p_2) + G_{\phi_1}(p_2) - G_{\phi_2}(p_2)$$

and deduce the following estimate

$$\begin{aligned} &|A^{-\gamma}(G_{\phi_1}(p_1) - G_{\phi_2}(p_2))|_\alpha \\ &\quad \leqslant L_F\{(1+l)\,|p_1 - p_2|_\alpha + \|\phi_1 - \phi_2\|_\alpha (1+|p_2|_\alpha)\}. \end{aligned} \tag{2.40}$$

Hence by (2.29) and (2.38) with $p_1^0 = p_2^0 = p^0$,

$$|A^{-\gamma}(G_{\phi_1}(p_1^{-k}) - G_{\phi_2}(p_2^{-k}))|_\alpha \leqslant L_F(1+k\beta)\,\eta^k\, \|\phi_1 - \phi_2\|_\alpha (1+|p^0|_\alpha).$$

We return to (2.39) and make use of (2.31) together with the last estimate. We find

$$\|\mathscr{T}_\tau\phi_1 - \mathscr{T}_\tau\phi_2\|_\alpha \leqslant \bar{\omega}\, \|\phi_1 - \phi_2\|_\alpha,$$

$$\bar{\omega} = \tau L_F \sum_{k=1}^{\infty} (A^\gamma + \tau^{-\gamma}k^{-\gamma})(1+\tau A)^{-k}\eta^k(1+k\beta). \tag{2.41}$$

Comparing with (2.33), we have

$$\bar{\omega} = \omega(1+l)^{-1} + \tau L_F \beta \sum_{k=1}^{\infty} k(A^\gamma + \tau^{-\gamma}k^{-\gamma})\, r^k, \tag{2.42}$$

$r = \eta(1+\tau A)^{-1}$. Using (2.34) and $\sum_{k=1}^{\infty} kr^k = r(1-r^2)^{-1}$, we find

$$\bar{\omega} \leqslant l(1+l)^{-1} + \tau L_F \beta(rA^\gamma(1-r^2)^{-1} + \tau^{-\alpha} I)$$

with $I = \sum_{k=1}^{\infty} k^{-(1-\gamma)} r^k \leqslant \int_0^{+\infty} x^{-(1-\gamma)} r^x \, dx \leqslant |\text{Log}\, r|^{\gamma-2} \leqslant (1-r)^{\gamma-2}$. Hence, making use of the fact that $\tau\Lambda \leqslant 1$,

$$\omega \leqslant l(1+l)^{-1} + \tau L_F \beta \eta^{-1} \Lambda^{\gamma}(r(1-r^2)^{-1} + (1-r)^{-2}).$$

Then, reporting the value of r and thanks to the inequalities $r/(1+r) \leqslant \frac{1}{2}$, $\tau L_F(1+l)\,\Lambda^{\gamma} \leqslant l(1-r)$, $\beta\eta^{-1} \leqslant 1$ and $\tau\Lambda \leqslant 1$, we find

$$\bar{\omega} \leqslant l(1+l)^{-1} + 2l. \tag{2.43}$$

This achieves the proof of Proposition 2.1. Concerning that of Lemma 2.3, we deduce from (2.13) and (2.18)

$$R(\tau)^{-1}(\delta p)_{n+1} = (\delta p)_n + P(F(p_1^n + \phi_1(p_1^n)) - F(p_2^n + \phi_2(p_2^n))). \tag{2.44}$$

We bound in $D(A^{\alpha})$ the term involving F by

$$\lambda^{\gamma} L_F((1+l)\,|(\delta p)_n|_{\alpha} + \|\phi_1 - \phi_2\|\,(1 + |p_1^n|_{\alpha})),$$

and then the scalar product of (2.44) with $(\delta p)_n$ leads to

$$\begin{aligned} &|(\delta p)_n|_{\alpha}(1 - \tau\lambda^{\gamma} L_F(1+l) \\ &\quad \leqslant (1+\tau\lambda)\,|(\delta p)_{n+1}|_{\alpha} + \tau\lambda^{\gamma} L_F\,\|\phi_1 - \phi_2\|_{\alpha}(1 + |p_1^n|_{\alpha}). \end{aligned}$$

Thanks to Lemma 2.2,

$$1 + |p_1^n|_{\alpha} \leqslant \eta^{-n}(1 + |p_1^0|_{\alpha}) + (\eta^{-n} - 1)(\beta(\eta-1)^{-1} - 1),$$

and since $\beta(\eta-1)^{-1} \leqslant 1$ we conclude that

$$|(\delta p)_n|_{\alpha} \leqslant \eta\,|(\delta p)_{n+1}|_{\alpha} + \beta\eta^{-n}(1+l)^{-1}\,\|\phi_1 - \phi_2\|_{\alpha}(1 + |p_1^0|_{\alpha}).$$

Finally (2.29) follows by induction on n.

2.1d. *The Discrete Cone-Property.* Let us first study the behavior of the difference of two solutions to the discretized Eq. (2.1). We take u_1^0 and u_2^0 in $D(A^{\alpha})$ and set $w^n = u_1^n - u_2^n$. Our aim is to estimate w^n, $n \geqslant 0$, in terms of w^0 and n. According to (2.1)

$$R(\tau)^{-1} w^{n+1} = w^n - \tau(F(u_1^n) - F(u_2^n)). \tag{2.45}$$

The scalar product of (2.45) with w^{n+1} then leads to

$$|w^{n+1}|_{\alpha}^2 + \tau\,|A^{1/2} w^{n+1}|_{\alpha}^2 \leqslant |w^{n+1}|_{\alpha}\,|w^n|_{\alpha} + \tau L_F\,|A^{\alpha} w^{n+1}|_{\alpha}\,|w^n|_{\alpha}, \tag{2.46}$$

hence using that $|A^{\gamma} w^{n+1}|_{\alpha} \leqslant \Lambda_1^{\gamma-1/2}\,|A^{1/2} w^{n+1}|_{\alpha}$, we find

$$|w^{n+1}|_{\alpha}^2 \leqslant |w^n|_{\alpha}\,|w^{n+1}|_{\alpha} + \tau(L_F \Lambda_1^{\gamma-1/2}\,|w^n|_{\alpha}\,|A^{1/2} w^{n+1}|_{\alpha} - |A^{1/2} w^{n+1}|_{\alpha}^2).$$

It follows that

$$|w^{n+1}|_\alpha^2 \leqslant (1 + \tau L_F^2 \Lambda_1^{2\gamma-1}/2)\, |w^n|_\alpha^2, \qquad \forall n \geqslant 0,$$

where we have used the inequality $x^2 - L_F \Lambda_1^{\gamma-1/2} xy \geqslant -L_F^2 \Lambda_1^{2\gamma-1} y^2/4$. This shows that

$$|u_1^n - u_2^n|_\alpha \leqslant (1 + \tau L_F^2 \Lambda_1^{2\gamma-1}/\tau)^{n-2}\, |u_1^0 - u_2^0|_\alpha, \qquad \forall n \geqslant 0. \tag{2.47}$$

This estimate allows exponential divergence of discrete trajectories which is a general feature for trajectories of the dynamical systems we consider in this work. However, as we shall see later in Proposition 2.2, a couple of trajectories that diverge must lie inside a cone in $D(A^\alpha)$ after a dinite transcient. Moreover those which do not are squeezed. This kind of result in the context of infinite dimensional dynamical systems was introduced by Foias–Temam [12] in their study on the Navier–Stokes equation. The more accomplished result in the case of continuous dynamical systems, referred to as the cone property in the first section, is due to Foias, Nicolaenko, Sell, and Temam [9]. In the discrete case, we prove

PROPOSITION 2.2. *We are given $\kappa > 0$ and assume that N satisfies*

$$\Lambda_{N+1} - \Lambda_N \geqslant 4L_F((1 + \kappa^{-1})\, \Lambda_N^\gamma + (1 + \kappa)\, \Lambda_{N+1}^\gamma). \tag{2.48}$$

Then for every τ satisfying (2.20), *the cone $\mathscr{C}_\kappa$ in* (1.26), *where $P = P_N$, $Q = I - P_N$ is invariant by* (2.1), *i.e.,*

$$\textit{if } u_1^0 - u_2^0 \in \mathscr{C}_\kappa \textit{ then } u_1^n - u_2^n \in \mathscr{C}_\kappa, \qquad \forall n \geqslant 0. \tag{2.49}$$

Moreover, we have the following alternative: either

$$\exists k \geqslant 0, \qquad u_1^k - u_2^k \in \mathscr{C}_\kappa \tag{2.50}$$

or

$$\begin{gathered} |u_1^n - u_2^n|_\alpha \leqslant (1 + 1/\kappa)\, \rho^n\, |u_1^0 - u_2^0|_\alpha, \qquad \forall n \geqslant 0, \\ \rho = (1 + \tau(1 + \kappa)\, L_F \Lambda_{N+1}^\gamma)(1 + \tau \Lambda_{N+1})^{-1} < 1. \end{gathered} \tag{2.51}$$

Proof. We begin by (2.49). For we are given $u_1^0, u_2^0 \in D(A^\alpha)$ with $u_1^0 - u_2^0 \in \mathscr{C}_\kappa$, i.e., $|p^0|_\alpha \geqslant \kappa\, |q^0|_\alpha$, where $p^0 = P(u_1^0 - u_2^0)$, $q^0 = Q(u_1^0 - u_2^0)$. Our aim is to show that

$$|p^1|_\alpha \geqslant \kappa\, |q^1|_\alpha, \tag{2.52}$$

where $p^1 = P(u_1^1 - u_2^1)$, $q^1 = Q(u_1^1 - u_2^1)$. Using (2.1), we have

$$R(\tau)^{-1} p^1 = p^0 - \tau P(F(u_1^0) - F(u_2^0)).$$

The scalar product of this relation with p^0 in $D(A^\alpha)$ gives

$$(1+\tau\lambda)\,|p^1|_\alpha\,|p^0|_\alpha \geqslant |p^0|_\alpha^2 - \tau\,|A^{-\gamma}P(F(u_1^0)-F(u_2^0))|_\alpha\,|A^\gamma p^0|_\alpha,$$

and using (1.6) we find

$$(1+\tau\lambda)\,|p^1|_\alpha\,|p^0|_\alpha \geqslant |p^0|_\alpha^2 - \tau L_F\lambda^\gamma\,|u_1^0-u_2^0|_\alpha\,|p^0|_\alpha.$$

Now since $|p^0|_\alpha \geqslant \kappa\,|q^0|_\alpha$ it follows that

$$|p^0|_\alpha \leqslant \frac{1+\tau\lambda}{1-\tau L_F(1+1/\kappa)\,\lambda^\alpha}\,|p^1|_\alpha \equiv \eta_\kappa\,|A^\alpha p^1|, \tag{2.53}$$

and we note that (2.20) and (2.48) implies that $\eta_\kappa \leqslant 4$. Concerning the components in $QD(A^\alpha)$, we return to (2.1) and deduce

$$R(\tau)q^1 = q^0 - \tau Q(F(u_1^0)-F(u_2^0)).$$

The scalar product with q^1 in $D(A^\alpha)$ leads to

$$|q^1|_\alpha^2 + \tau\,|A^{1/2}q^1|_\alpha^2 \leqslant |q^0|_\alpha\,|q^1|_\alpha + \tau L_F\,|u_1^0-u_2^0|_\alpha\,\Lambda^{\gamma-1/2}\,|A^{1/2}q^1|_\alpha.$$

Then using that $|p^0|_\alpha \geqslant \kappa\,|q^0|_\alpha$ and (2.53) we obtain

$$\begin{aligned}&|q^1|_\alpha^2 + \tau\,|A^{1/2}q^1|_\alpha^2\\ &\quad\leqslant |q^0|_\alpha\,|q^1|_\alpha + \tau L_F(1+1/\kappa)\eta_\kappa\Lambda^{\gamma-1/2}\,|A^{1/2}q^1|_\alpha\,|p^1|_\alpha.\end{aligned} \tag{2.54}$$

We claim that thanks to (2.20) and (2.48), this inequality implies (2.52). Indeed, if not, we have $|q^1|_\alpha \geqslant \kappa^{-1}\,|p^1|_\alpha$ thus

$$\begin{aligned}|A^{1/2}q^1|_\alpha &\geqslant \Lambda^{1/2}\,|q^1|_\alpha \geqslant \Lambda^{1/2}\kappa^{-1}\,|p^1|_\alpha\\ &\geqslant L_F(1+\kappa^{-1})\,\eta_\kappa\Lambda^{\gamma-1/2}\,|p^1|_\alpha/2 \qquad \text{(by (2.48) and } \eta_\kappa \leqslant 4).\end{aligned}$$

Then we can replace $|A^{1/2}q^1|_\alpha$ in (2.54) by $\Lambda^{1/2}\,|q^1|_\alpha$,

$$(1+\tau\Lambda)\,|q^1|_\alpha^2 \leqslant |q^0|_\alpha\,|q^1|_\alpha + \tau L_F(1+\kappa^{-1})\,\eta_\kappa\Lambda^\gamma\,|q^1|_\alpha\,|p^1|_\alpha.$$

Using that $\kappa\,|q^0|_\alpha \leqslant |p^0|_\alpha \leqslant |p^0|_\alpha \leqslant \eta_\kappa\,|p^1|_\alpha$ we conclude that

$$(1+\tau\Lambda)\,|q^1|_\alpha \leqslant (\eta_\kappa\kappa^{-1} + \tau L_F(1+\kappa^{-1})\,\eta_\kappa\Lambda^\alpha)\,|p^1|_\alpha.$$

Finally (2.20) and (2.48) implies (2.52) and the proof of (2.49) is achieved.

Concerning (2.51), we assume that $u_1^n - u_2^n \notin \mathscr{C}_\kappa$, $\forall n \geqslant 0$, i.e.,

$$|p^n|_\alpha \leqslant \kappa\,|q^n|_\alpha, \qquad p^n = P(u_1^n-u_2^n),\ q^n = Q(u_1^n-u_2^n), \qquad \forall n \geqslant 0. \tag{2.55}$$

We are going to prove that

$$|q^{n+1}|_\alpha \leqslant \rho\, |q^n|_\alpha, \qquad \forall n \geqslant 0, \tag{2.56}$$

which shows (2.51) according to (2.55). The scalar product of (2.1) with q^{n+1} leads to

$$\begin{aligned} &|q^{n+1}|_\alpha^2 + |A^{1/2} q^{n+1}|_\alpha^2 \\ &\quad \leqslant |q^n|_\alpha\, |q^{n+1}|_\alpha + \tau L_F\, |u_1^n - u_2^n|_\alpha\, |A^\gamma q^{n+1}|_\alpha, \\ &|q^{n+1}|_\alpha^2 + |A^{1/2} q^{n+1}|_\alpha^2 \\ &\quad \leqslant |q^n|_\alpha\, |q^{n+1}|_\alpha + \tau L_F (1+\kappa)\, \Lambda^{\gamma - 1/2}\, |q^n|_\alpha\, |q^{n+1}|_\alpha. \end{aligned} \tag{2.57}$$

If (2.56) does not hold, then

$$\begin{aligned} |A^{1/2} q^{n+1}|_\alpha &\geqslant \Lambda^{1/2}\, |q^{n+1}|_\alpha \geqslant \rho \Lambda^{1/2}\, |q^n|_\alpha \\ &\geqslant L_F (1+\kappa)\, \Lambda^{\alpha - 1/2}\, |q^n|_\alpha / 2, \qquad \text{(by (2.20) and (2.48))} \end{aligned}$$

and we can replace $|A^{1/2} q^{n+1}|_\alpha$ by $\Lambda^{1/2}\, |q^{n+1}|_\alpha$ in (2.57):

$$(1 + \tau\Lambda)\, |q^{n+1}|_\alpha \leqslant |q^n|_\alpha (1 + \tau L_F (1+\kappa)\, \Lambda^\alpha).$$

This is exactly (2.56).

2.1e. *The Proof of Theorem* 2.1. We label the hypotheses on N in Theorem 2.1 as follows

$$\Lambda_{N+1} \geqslant 3 L_F^2 \Lambda_1^{2\gamma - 1}/2, \tag{2.57'}$$

$$\Lambda_{N+1} - \Lambda_N \geqslant 30 L_F (\Lambda_N^\gamma + \Lambda_{N+1}^\gamma). \tag{2.58}$$

Let us take $l = \frac{1}{4}$ and $\kappa = 4$, then (2.58) implies (2.19) and (2.48) and the conclusions of Corollary 2.1 and Proposition 2.2 hold true. Hence we have found a fixed point ϕ_τ of $\mathscr{T}_\tau$ in $\mathscr{F}_{1/4}$ and (2.50), (2.51) are satisfied for $\mathscr{C}_4$. In order to achieve the proof of Theorem 2.1, we have to show the exponential attraction property (which is stronger than (2.6)).

For, we take $u^0 \in D(A^\alpha)$ and denote by $v^0 \in M_\tau$ a point such that $d_\alpha(u^0, M_\tau) = |u^0 - v^0|_\alpha$. We set $u^n = (S_{\phi_\tau}^\tau)^n u^0$ and $v^n = (S_{\phi_\tau}^\tau) v^0 = Pv^n + \phi_\tau(Pv^n)$. We first note that (2.57), (2.58) and the fact that $\tau\Lambda \leqslant 1$ leads (after some tedious calculation) to

$$(1 + \tau L_F^2 \Lambda_1^{2\gamma - 1}/2)(1 + (1+\kappa) L_F \Lambda^\gamma) \leqslant (1 + \tau\Lambda), \qquad \kappa = 4. \tag{2.59}$$

We set $n_0 = 1 + [(2 \operatorname{Log}(1 + \tau L_F^2 \Lambda_1^{2\gamma-1}/2))^{-1} \operatorname{Log} 2]$, where $[x]$ denotes the integer part of x. Then, due to (2.57),

$$(1 + \tau L_F^2 \Lambda_1^{2\gamma-1}/2)^{n_0} \leqslant \exp(\operatorname{Log} 2^{1/2} (1 + \tau L_F^2 \Lambda_1^{2\gamma-1}/2)) \leqslant 4\sqrt{2}/3, \tag{2.60}$$

where we have used that $\tau L_F^2 \Lambda_1^{2\gamma-1}/2 \leqslant \tau\Lambda/2 \leqslant \frac{1}{2}$. On the other hand ($\kappa = 4$), thanks to (2.59) and the definition of n_0,

$$(1+(1+\kappa)\,\tau L_F \Lambda^\gamma)^{2n_0}(1+\tau\Lambda)^{-2n_0} \leqslant (1+\tau L_F^2 \Lambda_1^{2\gamma-1}/2)^{-2n_0} \leqslant \tfrac{1}{2}. \quad (2.61)$$

Second, we take $n \in \mathbb{N}$ satisfying $n_0 \leqslant n \leqslant 2n_0$. Two cases can occur.

(i) Either, $u^n - v^n \in \mathscr{C}_4$ and then

$$\begin{aligned} d_\alpha(u^n, M_\tau) &\leqslant (\text{since } Pu^n + \phi_\tau(Pu^n) \in M_\tau) \\ &\leqslant |Qu^n - \phi_\tau(Pu^n)|_\alpha \\ &\leqslant |Q(u^n - v^n)|_\alpha + |\phi_\tau(Pu^n) - \phi_\tau(Pv^n)|_\alpha \\ &\leqslant (\text{since } u^n - v^n \in \mathscr{C}_4 \text{ and } \phi_\tau \in \mathscr{F}_{1/4}) \\ &\leqslant |u^n - v^n|_\alpha/2 \leqslant (\text{by (2.47) and } n \leqslant 2n_0) \\ &\leqslant |u^n - v^n|_\alpha (1+\tau L_F^2 \Lambda_1^{2\gamma-1}/2)^{n_0}/2 \\ &\leqslant 2\sqrt{2}\,|u^n - v^n|_\alpha/3. \end{aligned}$$

Hence

$$n_0 \leqslant n \leqslant 2n_0,\; u^n - v^n \in \mathscr{C}_4 \Rightarrow d_\alpha(u^n, M_\tau) \leqslant (2\sqrt{2}/3)\, d_\alpha(u^0, M_\tau). \quad (2.62)$$

(ii) Or, $u^n - v^n \notin \mathscr{C}_4$ and then by Proposition 2.2, $u^k - v^k \notin \mathscr{C}_4$ for $k = 0, ..., n_0$. We can use (2.51), with $u_1^k = u^k$ and $u_2^k = v^k$. This leads to ($\kappa = 4$)

$$\begin{aligned} d_\alpha(u^n, M_\tau) \leqslant |u^n - v^n|_\alpha &\leqslant (1+1/4)(1+5\tau L_F \Lambda^\gamma)^{2n_0}(1+\tau\Lambda)^{-2n_0}\,|u^0 - v^0|_\alpha \\ &\leqslant (5/8)\,|u^0 - v^0|_\alpha \qquad (\text{by (2.61)}). \end{aligned}$$

Here,

$$n_0 \leqslant n \leqslant 2n_0\;\; u^n - v^n \notin \mathscr{C}_4 \Rightarrow d_\alpha(u^n, M_\tau) \leqslant (5/4\sqrt{2})\, d_\alpha(u^0_{n'} M_\tau).$$

We summarize this and (2.62) in

$$d_\alpha(u^n, M_\tau) \leqslant (2\sqrt{2}/3)\, d_\alpha(u^0, M_\tau) \qquad \text{for} \quad n_0 \leqslant n \leqslant 2n_0. \quad (2.63)$$

Now if $n \geqslant 2n_0$, we introduce $k \in \mathbb{N}$ ad $r \in [n_0 + 1, 2n_0]$ such that $n = k_{n_0} + r$. According to (2.64), we have

$$d_\alpha(u^n, M_\tau) \leqslant (2\sqrt{2}/3)^k\, d_\alpha(u^0, M_\tau), \qquad n \geqslant n_0$$

and this shows the exponential attraction of the inertial manifold with $\sigma = [\mathrm{Log}(3/2\sqrt{2})]\, n_0$. ∎

2.2. *The General Case*

In this section we consider the general case, i.e., the case where $C \neq 0$. A lot of schemes are available in order to discretize the evolution equation (1.4), and a choice similar to (2.1) would be

$$(u^{n+1} - u^n)/\tau + (A + C)u^{n+1} + F(u^n) = 0, \qquad n \geqslant 0. \tag{2.64}$$

This is not in fact a good method since our motivation is the study of the behavior of (1.4) on very long time intervals. For example, when $F = 0$, (2.64) will dissipate too much "energy" with respect to the continuous case $du/dt + (A + C)u = 0$. Indeed in this last equation, since C is squew-symmetric, this operator does not contribute to the value of the norms $|A^s u(t)|$, $s \in \mathbb{R}$. While in (2.64) it is the case. On the other hand, if A were not present and $F = 0$, it is well known that leap-frog schemes, like, e.g.,

$$(u^{n+1} - u^{n-1})/2\tau + (A + C)(u^{n+1} + u^{n-1})/2\tau + F(u^n) = 0, \tag{2.65}$$

are energy preserving. This choice is not good either since in that case it is the dissipation of A which is not well approximated (on long time interval).

In order to represent the dynamics of (1.4) on an arbitrarily large time interval, we propose combining the two schemes (2.64) and (2.65) through a *fractional-step-method*:

$$\begin{aligned} &\frac{u^{n+1/2} - u^n}{\tau} + Au^{n+1/2} + F(u^n) = 0, \\ &\frac{u^{n+1} - u^{n+1/2}}{\tau} + C\frac{u^{n+1} + u^{n+1/2}}{2} = 0. \end{aligned} \tag{2.66}$$

We note that, when $C = 0$, we recover the classical semi-implicit scheme we have used in the previous sections.

2.2a. *A Fractional Step Method.* As for the case where $C = 0$, we rewrite (2.66), as

$$u^{n+1/2} = R(\tau)(u^n - \tau F(u^n)), \tag{2.67}$$

with $R(\tau) = (I + \tau A)^{-1}$. Concerning $(2.66)_2$, we use the notations of Section 1 and set

$$U(\tau)v = \sum_{\Lambda \in \sigma(A)} U_\Lambda(\tau)\, R_\Lambda v, \tag{2.68}$$

where $U_\Lambda(\tau)$ is the unitary operator on $R_\Lambda H$: $U_\Lambda(\tau) = (I + \tau C_\Lambda/2)^{-1}$ $(I - \tau C_\Lambda/2)$. The $U(\tau)$ are unitary on the $D(A^s)$, $s \in \mathbb{R}$ and $(2.66)_2$ reads

$$u^{n+1} = U(\tau)\, u^{n+1/2}. \tag{2.69}$$

In conclusion (2.66) can be written as

$$u^{n+1} = \bar{R}(\tau)(u^n - \tau F(u^n)), \qquad n \geqslant 0, \tag{2.70}$$

where

$$\bar{R}(\tau) \equiv U(\tau)\, R(\tau) = R(\tau)\, U(\tau), \qquad \forall \tau > 0. \tag{2.71}$$

In other words, by comparison with (2.3), we have replaced $R(\tau)$ by $\bar{R}(\tau)$ given above.

2.2b. *Existence of an Inertial Manifold.* The analogue of Theorem 2.1 is

THEOREM 2.2. *We assume that $N \geqslant 1$ is such that*

$$\Lambda_{N+1} \geqslant 3L_F^2 \Lambda_1^{2\gamma-1}/2,$$
$$\Lambda_{N+1} - \Lambda_N \geqslant 30 L_F(\Lambda_N^\gamma + \Lambda_{N+1}^\gamma),$$

then for every $\tau > 0$ such that $\tau \Lambda_{N+1} \leqslant 1$, the discrete infinite dimensional dynamical system

$$\frac{u^{n+1/2} - u^n}{\tau} + A u^{n+1/2} + F(u^n) = 0,$$
$$\frac{u^{n+1} - u^{n+1/2}}{\tau} + C\frac{u^{n+1} + u^{n+1/2}}{2} = 0,$$

possesses an inertial manifold M_τ which is the graph of a Lipschitz function from $P_N H$ into $Q_N D(A^\alpha)$. Moreover there exists two constants C_0 and $\sigma > 0$ such that for every $u^0 \in D(A^\alpha)$,

$$d_\alpha(u^n, M_\tau) \leqslant C_0\, e^{-\sigma n}\, d_\alpha(u^0, M_\tau), \qquad \forall n \geqslant 0. \tag{2.72}$$

The proof of this result is mutatis mutandis that of Theorem 2.1 after replacing $R(\tau)$ by $\bar{R}(\tau)$. Indeed the properties of $R(\tau)$ we have used in Section 2.1 remain true for $\bar{R}(\tau)$ since $U(\tau)$ is unitary and commutes to A. This is not surprising since it is also the case in the continuous case [5, Sect. 1], where exp At is replaced by $\exp(A + C)t$.

3. CONVERGENCE OF THE APPROXIMATE INERTIAL MANIFOLDS

Assuming that N is such that

$$\begin{aligned} \Lambda_{N+1} &\geqslant 3L_F^2 \Lambda_1^{2\gamma-1}/2, \\ \Lambda_{N+1} - \Lambda_N &\geqslant 30 L_F(\Lambda_N^\gamma + \Lambda_{N+1}^\gamma), \end{aligned} \tag{3.0}$$

we know according to Theorem 1.1, that the continuous Eq. (1.4) possesses an M-dimensional ($M = \dim P_N H$) inertial manifold M. For sufficiently small τ, i.e., $\tau \Lambda_{N+1} \leqslant 1$, Theorem 2.2 provides an M-dimensional inertial manifold M_τ for the discretized Eq. (2.66). A natural question is whether M_τ converges to M as τ goes to zero. This result is expected since these objects are obtained via the Banach fixed point theorem, a robust method. However, as we shall see in Section 3.2, the control of the infinite dimensional parts, i.e., in $Q_N H$, is slightly delicate since we are dealing with unbounded operators.

We denote by ϕ the equation of M and by ϕ_τ that of M_τ. The convergence result is then

THEOREM 3.1. *We assume that N satisfies* (3.0) *and that* $\tau \Lambda_{N+1} \leqslant 1$. *There exists a constant K which is independent of $\tau > 0$ such that*

$$\|\phi - \phi_\tau\|_\alpha \leqslant K\tau^\zeta (1 + |\text{Log}\, \tau \Lambda_{N+1}|)^\varepsilon,$$

where

$$\zeta = 1, \qquad \varepsilon = 1 \quad \textit{for} \quad s_0 \leqslant 1 \textit{ and } \gamma = 0$$

$$\zeta = 1 - \gamma, \qquad \varepsilon = 0 \quad \textit{for} \quad s_0 \leqslant 1 \textit{ and } \gamma > 0,$$

$$\zeta = (1-\gamma)(2s_0 - 1)^{-1}, \qquad \varepsilon = 1 \quad \textit{for} \quad s_0 > 1.$$

The proof of this result is given in Section 3.3. The two following sections are devoted to error estimates for arbitrary $\phi \in \mathscr{F}_l$. The first one deals with the finite dimensional part: we estimate the difference between $S_\phi(n\tau)$ and (S_ϕ^τ) on $P_N H$. The second one concerns the infinite dimensional part: we study $\mathscr{T}\phi - \mathscr{T}_\tau \phi$.

3.1. *An Error Estimate on the Finite Dimensional Part*

We are given $\phi \in \mathscr{F}_l$, and our aim is to relate the two following dynamical systems:

$$\frac{dp}{dt} + Ap + Cp + PF(p + \phi(p)) = 0, \tag{3.1}$$

$$p^{n+1} = \bar{R}(\tau)(p^n - \tau PF(p^n + \phi(p^n))), \tag{3.2}$$

where $P = P_N$, τ satisfy (2.10), $\bar{R}(\tau) = R(\tau)\, U(\tau) = (I + \tau A)^{-1}(I + \tau C/2)^{-1}$ $(I - \tau C/2)$ and N is a fixed integer. We do not assume that ϕ is the graph of an inertial manifold.

Since (3.1) is a standard ODE, we know from classical results on one step methods that the discretization error

$$e_n = p(n\tau) - p^n, \qquad n \in \mathbb{Z} \tag{3.3}$$

tends to zero with τ, like a power of τ in general: the order of the method. In our case we have the following estimate.

PROPOSITION 3.1. *With the previous hypotheses and notations, let p^0 be given in H and let $p(t)$, $t \in \mathbb{R}$ (resp. $p^n, n \in \mathbb{Z}$), denote the solution to* (3.1) *(resp.* (3.2)*) satisfying $p(0) = p^0$ (resp. $p_0 = p^0$). For every negative integer n, we have*

$$|e_n|_\alpha \leqslant \frac{\tau^2 K(\lambda)}{1 - \tau L_F(1+l)\,\lambda^\alpha}\,(\eta^{-n} - e^{-n\tau\bar{\lambda}})(\eta e^{-\tau\bar{\lambda}} - 1)(1 + |p^0|_\alpha), \qquad (3.4)$$

where $K(\lambda)$ is independent of τ and n, $\eta = (1+\tau\lambda)(1 - \tau L_F(1+l)\,\lambda^\gamma)^{-1}$, $\bar{\lambda} = \lambda + \lambda^\gamma L_F(1+l)$, and $\lambda = \Lambda_N$.

Proof. This is a very classical matter, and we only briefly sketch the main steps of the proof. We introduce the consistency error, ε_n, by writing

$$\bar{R}(\tau)^{-1} p((n+1)\tau) = (p^n - \tau PF(p(n\tau) + \phi(p(n\tau))) + \varepsilon_n. \qquad (3.5)$$

By comparison with (3.2), we have

$$\bar{R}(\tau)^{-1}\, e_{n+1} = (e_n - \tau P(G(p(n\tau)) - G(p^n)) + \varepsilon_n, \qquad (3.6)$$

where $G(p) = GF(p + \phi(p))$. The scalar product of (3.6) with e_n in $D(A^\alpha)$ leads then to

$$|e_n|_\alpha \leqslant \eta\; |e_{n+1}|_\alpha + |\varepsilon_n|_\alpha / (1 - \tau(1 - \tau(1+l)\, L_F \lambda^\alpha). \qquad (3.7)$$

This shows that $(e_0 = 0)$

$$|e_n|_\alpha \leqslant \left(\sum_{k=n}^{-1} \eta^{k-n}\, |\varepsilon_k|_\alpha\right)(1 - \tau L_F(1+l))^{-1}, \qquad \forall n \leqslant 0. \qquad (3.8)$$

It remains to estimate the ε_k. We introduce the function $\varphi(s) = \bar{R}(s)^{-1} p(\sigma + s) - p(\sigma) + sPF(p(\sigma) + \phi(p(\sigma)))$, where σ is fixed and $s \geqslant 0$. We have

$$\varepsilon_n = \varphi(\tau), \qquad \sigma = n\tau, \qquad (3.9)$$

and since $\varphi(0) = 0$, there exists $\theta \in\,]0, 1[$ such that

$$|\varepsilon_n|_\alpha \leqslant \tau\; |\varphi'(\theta\tau)|_\alpha, \qquad \sigma = n\tau. \qquad (3.10)$$

We compute $\varphi'(s)$,

$$\varphi'(s) = \left(\frac{d}{ds}\,\bar{R}(s)^{-1}\right) p(\sigma + s) - (A + C)\, p(\sigma) + \bar{R}(s)^{-1}\,\frac{dp}{ds}(\sigma + s) - \frac{dp}{ds}(\sigma),$$

that we rewrite as

$$\varphi'(s) = \psi_1(s) + \psi_2(s), \tag{3.11}$$

with $\psi_2(s) \equiv \bar{R}(s)^{-1}((dp/ds)(\sigma+s) - (dp/ds)(\sigma))$. The function ψ_1 is differentiable and since $\psi_1(0) = 0$, there exists $\theta_1 \in]0, 1[$ such that

$$\psi_1(s) = s\psi_1'(\theta_1 s). \tag{3.12}$$

Concerning ψ_2, we write

$$\frac{dp}{ds}(\sigma) - \frac{dp}{ds}(\sigma+s)$$
$$= (A + C)(p(\sigma+s) - p(\sigma)) + P(G(p(\sigma+s)) - G(p(\sigma))),$$

which shows that

$$\left|\frac{dp}{dt}(\sigma+s) - \frac{dp}{dt}(\sigma)\right|_\alpha \leqslant (\lambda + \lambda^{s_0} + (1+l)\, L_F \lambda^\alpha)\, |p(\sigma+s) - p(\sigma)|_\alpha,$$

where we have denoted

$$a = |C|_{\mathscr{L}(D(A^{s_0}), H)}. \tag{3.13}$$

Therefore there exists $\theta_2 \in]0, 1[$ such that (we use that $|\bar{R}(\tau)^{-1}|_{\mathscr{L}(D(A^\alpha))} \leqslant 1 + s\lambda$)

$$|\psi_2(s)|_\alpha \leqslant sA + s\lambda(\lambda + a\lambda^{s_0} + (1+l)\, L_F \lambda^\alpha) \left|\frac{dp}{dt}(\sigma + \theta_2 s)\right|_\alpha. \tag{3.14}$$

In order to derive from (3.12) and (3.14) an estimate on ε_n by using (3.10), we need an estimate on $|p(\sigma)|_\alpha$ for negative σ. Returning to (3.1) and thanks to (1.6)–(1.7), we find ($\bar{\lambda} = \lambda + \lambda^\gamma L_F(1+l)$)

$$|p(\sigma)|_\alpha \leqslant (1 + |p^0|_\alpha)\, e^{-\bar{\lambda}\sigma}, \qquad \forall \sigma \leqslant 0. \tag{3.15}$$

Then using (3.1) and (3.15), (3.14) leads to

$$|\psi_2(s)|_\alpha \leqslant s(1 + s\lambda)\, K_1 (1 + |p^0|_\alpha)\, e^{-\bar{\lambda}(s+n\tau)}, \qquad \forall s \leqslant 0, \tag{3.16}$$

where K_1 (and K_2 below) are constants which only depend on λ, L_F, l, s_0, and a. The estimate of (3.12) is similar. One obtains

$$|\psi_1(s)|_\alpha \leqslant s(2\lambda + (1 + s\lambda)\, a\lambda^{s_0})\, K_2 (1 + |p^0|_\alpha)\, e^{-\bar{\lambda}s}, \qquad \forall s \leqslant 0. \tag{3.17}$$

We sum (3.16) and (3.17), take $\sigma = n\tau \leqslant 0$, $s = \tau$ and regarding (3.10) we find

$$|\varepsilon_n|_\alpha \leqslant \tau^2 K(\lambda)(1 + |p^0|_\alpha)\, e^{-\lambda n\tau}, \qquad \forall n \leqslant 0. \tag{3.18}$$

Finally, (3.4) follows from (3.8) and (3.18).

3.2. *Convergence of the Mappings* $\mathscr{T}_\tau$ *to* $\mathscr{T}$

In this part, we take $\phi \in \mathscr{F}_l$ and our aim is to show that the $\mathscr{T}_\tau \phi$ converge to $\mathscr{T}\phi$ as $\tau \to 0$. We prove the following result

PROPOSITION 3.2. *The hypotheses are the same as in Proposition* 3.1. *We assume moreover that N is such that*

$$\Lambda_{N+1} - \Lambda_N \geqslant 2L_F(1 + l)\, \Lambda_N^\gamma. \tag{3.19}$$

For every $\tau > 0$, *such that* $\tau\Lambda_{N+1} \leqslant 1$, *and for every* $p^0 \in PH$, $\phi \in \mathscr{F}_l$ *we have*

$$|(\mathscr{T}_\tau\phi - \phi)(p^0)|_\alpha \leqslant C_0(1 + |p^0|_\alpha)\, \tau^\zeta (1 + |\mathrm{Log}\, \tau\Lambda_{N+1}|)^\varepsilon, \tag{3.20}$$

where the constant C_0 *depends only on* Λ_N, $\Lambda_{N+1} s_0$ *and* γ *but not on* τ; ζ *and* $\varepsilon \in \{0, 1\}$ *are as follows*

$$\zeta = 1, \qquad \varepsilon = 1 \qquad \textit{for} \quad \gamma = 0 \ \textit{and}\ 0 < s_0 \leqslant 1, \tag{3.21}$$

$$\zeta = 1 - \gamma, \qquad \varepsilon = 0 \qquad \textit{for} \quad \gamma \in \,]0, 1[\ \textit{and}\ 0 < s_0 \leqslant 1, \tag{3.22}$$

$$\zeta = (1 - \gamma)(2s_0 - 1)^{-1}, \qquad \varepsilon = 1 \qquad \textit{for} \quad \gamma \in \,]0, 1[\ \textit{and}\ s_0 \geqslant 1. \tag{3.23}$$

Proof. Thanks to (1.24) and (2.18), we have

$$(\mathscr{T}_\tau\phi - \mathscr{T}\phi)(p^0) = \int_{-\infty}^0 e^{(A+C)\sigma} QG(p(\sigma))\, d\sigma - \tau \sum_{k=1}^\infty \bar{R}(\tau)^k QG(p^{-k}),$$

where, as before, $G(p) \equiv F(p + \phi(p))$ and we have considered $\bar{R}(\tau)$ instead of $R(\tau)$ since C may be different from 0. We split this expression as follows

$$(\mathscr{T}_\tau\phi - \tau\phi)(p^0) = \text{Eq. (3.25)} + \text{Eq. (3.26)} + \text{Eq. (3.27)}, \tag{3.24}$$

$$\sum_{k=1}^\infty \int_{-\kappa\tau-\tau}^{-\kappa\tau} e^{(A+C)\sigma} Q(G(p(\sigma)) - G(p^{-k}))\, d\sigma, \tag{3.25}$$

$$\sum_{k=1}^\infty \int_{-\kappa\tau-\tau}^{-\kappa\tau} (e^{(A+C)\sigma} - \bar{R}(\tau)^k)\, QG(p^{-k})\, d\sigma, \tag{3.26}$$

$$\int_{-\gamma}^0 e^{(A+C)\sigma} QG(p(\sigma))\, d\sigma. \tag{3.27}$$

We start with

$$|A^{\gamma} e^{A\sigma} Qv|_\alpha \leqslant (\Lambda^\gamma + (2\,|\sigma|^{-1})^\gamma)\, e^{\Lambda\sigma}\, |Qv|_\alpha, \qquad \forall v \in D(A^\alpha), \forall \sigma < 0, \tag{3.28}$$

whose proof is parallel to that of (2.31). Then thanks to (1.7), (3.15), and (3.28) it is easy to find a constant $\bar{C}$, which is independent of τ satisfying $\tau\Lambda \leqslant 1$, such that

$$|\text{Eq. (3.27)}|_\alpha \leqslant \bar{C}\tau^{1-\gamma}(1 + |p^0|_\alpha). \tag{3.29}$$

Now the proof is divided in two parts. First we estimate (3.25) using Proposition 3.1 and (3.28). Second we majorize (3.26).

(i) An estimate of (3.25). We write for $k \geqslant 0$, $-k\tau - \tau \leqslant \sigma \leqslant -k\tau$,

$$p(\sigma) - p^{-k} = p(\sigma) - p(-k\tau) + p(-k\tau) - p^{-k}$$

and we deduce from (3.1) and (3.15) that there exists $\theta \in\,]0, 1[$ such that

$$\begin{aligned} &|p(\sigma) - p^{-k}|_\alpha \leqslant \tau \left|\frac{dp}{dt}(-k\tau - \theta\tau)\right|_\alpha + |p(-k\tau) - p^{-k}|_\alpha \\ &|p(\sigma) - p^{-k}|_\alpha \leqslant \tau K_1 (1 + |p^0|_\alpha)(\exp \bar{\lambda}(k+1)\tau) + |p(-k\tau) - p^{-k}|_\alpha, \end{aligned} \tag{3.30}$$

where here and in the sequel K_i denotes constants which are independent of τ (satisfying $\tau\Lambda \leqslant 1$). Now using (3.4) with $n = -k$, we find

$$|p(-k\tau) - p^{-k}|_\alpha \leqslant \tau^2 K_2 (\eta^k - e^{k\bar{\lambda}\tau})(\eta e^{-\tau\bar{\lambda}} - 1)^{-1}(1 + |p^0|_\alpha).$$

Since $(\eta^k - e^{k\bar{\lambda}})(\eta e^{-\bar{\lambda}} - 1)^{-1} \leqslant k e^{\bar{\lambda}k}$, we conclude according to (3.30) that for $k \geqslant 0$

$$|p(\sigma) - p_{-k}|_\alpha \leqslant \tau(K_1 + k\tau K_2)(1 + |p^0|_\alpha)\, e^{\bar{\lambda}k\tau}, \qquad k\tau \leqslant -\sigma \leqslant (k+1)\tau. \tag{3.31}$$

Returning to (3.25), we set $v = A^{-\gamma}(G(p(\sigma)) - G(p^{-k}))$ and use that $|v|_\alpha \leqslant L_F(1 + l)\,|p(\sigma) - p^{-k}|_\alpha$. Then according to (3.28) we deduce, thanks to (3.31), that

$$\begin{aligned} |\text{Eq. (3.25)}|_\alpha \leqslant{}& \tau L_F(1 + l)(1 + |p^0|_\alpha) \\ &\times \int_{-\infty}^{0} (\Lambda + (2\,|\sigma|^{-1}))(K_1 + |\sigma|\, K_2) \exp(\Lambda - \bar{\lambda})\, \sigma\, d\sigma. \end{aligned} \tag{3.32}$$

We have by (3.19), $\Lambda - \bar{\lambda} \geqslant L_F(1 + l)\lambda^\gamma > 0$. Since $\gamma \in [0, 1[$, the integral above is convergent and we can find K_3 such that

$$|\text{Eq. (3.25)}|_\alpha \leqslant \tau K_3(1 + |p^0|_\alpha). \tag{3.33}$$

(ii) An estimate of (3.26). We split this sum as follows

$$\sum_{k=1}^{\infty} \int_{-k\tau-\tau}^{-k\tau} (e^{(A+C)\sigma} - e^{-(A+C)k\tau})\, QG(p^{-k})\, d\sigma, \tag{3.34}$$

$$\tau \sum_{k=1}^{\infty} (e^{-(A+C)k\tau} - \bar{R}(\tau)^k)\, QG(p^{-k}). \tag{3.35}$$

These two terms will be estimated by the same techniques. Let us first state two lemmas that we shall use for that purpose, postponing their proofs to the end of this section.

LEMMA 3.1. *There exists a constant C_1, which is independent of N and τ satisfying $\tau\Lambda \leqslant 1$, such that*

$$\sum_{k=1}^{\infty} e^{\bar{\lambda}k\tau} \int_{-k\tau-\tau}^{-k\tau} |A^{\gamma}(e^{(A+C)\sigma} - e^{-(A+C)k\tau})|_{\mathscr{L}(QD(A^{\alpha}))}\, d\sigma$$
$$\leqslant C_1 \tau^{\zeta_1}(1 + |\mathrm{Log}(\tau\Lambda|)^{\varepsilon_1} \tag{3.36}$$

where $\zeta_1 \in \,]0, 1[$, $\varepsilon_1 \in \{0, 1\}$ are given by the following table:

γ	s_0	ζ_1	ε_1
$\gamma = 0$	$0 < s_0 \leqslant 1$	1	1
$\gamma = 0$	$s_0 > 1$	$1/s_0$	1
$0 < \gamma < 1$	$0 < s_0 \leqslant 1$	$1 - \gamma$	0
$0 < \gamma < 1$	$s_0 > 1$	$(1-\gamma)/s_0$	1

LEMMA 3.2. *There exists a constant C_2, which is independent of N and τ satisfying $\tau\Lambda \leqslant 1$, such that*

$$\tau \sum_{k=1}^{\infty} |A^{\gamma}(e^{-(A+C)k\tau} - \bar{R}(\tau)^k|_{\mathscr{L}(QD(A^{\alpha}))}\, e^{\bar{\lambda}k\tau} \leqslant C_2 \tau^{\zeta_2}(1 + |\mathrm{Log}\, \tau\Lambda|)^{\varepsilon_2}, \tag{3.37}$$

where $\zeta_2 \in \,]0, 1[$, $\varepsilon_2 \in \{0, 1\}$ are

$\zeta_2 = 1 - \gamma$, $\quad \varepsilon_2 = 0 \quad$ *for* $\quad s_0 < 1$, $\varepsilon_2 = 1$ *for* $\gamma = 0$,

$\zeta_2 = (1-\gamma)(2s_0 - 1)^{-1}$, $\quad \varepsilon_2 = 1 \quad$ *for* $\quad s_0 \geqslant 1$.

Thanks to these lemmas, we are now able to estimate (3.34) and (3.35). Indeed using (1.7), (3.15), and (3.36), we find

$$|\mathrm{Eq.}\ (3.34)|_{\alpha} \leqslant K_4(1 + |p^0|_{\alpha})\, \tau^{\zeta_1}(1 + |\mathrm{Log}\, \tau\Lambda|)^{\varepsilon_1}. \tag{3.38}$$

In the same way, this time using (3.37), we have

$$|\text{Eq. } (3.35)|_\alpha \leqslant K_5(1+|p^0|_\alpha)\,\tau^{\zeta_2}(1+|\text{Log}\,\tau\Lambda|)^{\varepsilon_2}. \tag{3.39}$$

Combining (3.38) and (3.39), we deduce

$$\begin{gathered}|\text{Eq. } (3.26)|_\alpha \leqslant K_6(1+|p^0|_\alpha)\,\tau^{\zeta}(1+|\text{Log}\,\tau\Lambda|)^{\varepsilon},\\ \zeta=\min(\zeta_1,\zeta_2), \qquad \varepsilon=\max(\varepsilon_1,\varepsilon_2).\end{gathered} \tag{3.40}$$

Finally (3.20) follows by adding the estimates (3.29), (3.33), and (3.40).

Proof of Lemma 3.1. The norm of the operators A^γ, $e^{(A+C)\sigma}$, ... are easy to compute if we take an appropriate Hilbert basis of H. Indeed since A and C commute, we know that on each of the finite dimensional space $R_\mu H$, $\mu \in \sigma(A)$, the operator $R_\mu C$ is a squew-symmetric matrix. Using the complexification of H, it is possible to construct a Hilbert basis of H, $(e_p)_{p\in\mathbb{N}^*}$, such that

$$Ae_p=\mu_p e_p, \qquad Ce_p=i\xi_p e_p, \qquad \mu_p\in\mathbb{R}_+^*,\ \xi_p\in\mathbb{R}.$$

Moreover, we have the relation ($a=|C|_{\mathscr{L}(D(A^{s_0}),H)}$)

$$|\xi_p|\leqslant a\,|\mu_p|^{s_0}, \qquad \forall p\in\mathbb{N}^*.$$

Recall now that if B is a diagonal operator in the basis $(e_p)_{p\in\mathbb{N}^*}$, then

$$|B|_{\mathscr{L}(D(A^\alpha))}=\operatorname*{Sup}_{p\geqslant 1}|(Be_p,e_p)|.$$

Hence in order to estimate (3.36), we have to study for $k\geqslant 0$, $-\kappa\tau-\tau\leqslant\sigma\leqslant -k\tau$

$$\mu^\gamma(e^{(\mu+i\xi)\sigma}-e^{-(\mu+i\xi)k\tau}), \tag{3.41}$$

where $\mu\geqslant\Lambda$ and $|\xi|\leqslant a\,|\mu|^{s_0}$, $i\xi$ being an eigenvalue of C. We write

$$\text{Eq. } (3.41)=\mu^\gamma(e^{\mu\sigma}-e^{-\mu k\tau})^{i\xi\sigma}+\mu(e^{i\xi\sigma}-e^{-i\xi k\tau})\,e^{-\mu k\tau}$$

and majorize separately these two terms. First

$$\mu^\gamma\,|e^{\mu\sigma}-e^{-\mu k}|\leqslant\tau\mu^{1+\gamma}\,e^{-\mu k\tau}.$$

Second,

$$\begin{aligned}\mu^\gamma\,|e^{i\xi\sigma}-e^{-i\xi k\tau}|&=2\mu^\gamma\left|\sin\frac{\xi(\sigma-k\tau)}{2}\right|\\&\leqslant 2^{1-\delta_1}\mu^\gamma\,|\xi\tau|^{\delta_1},\end{aligned}$$

where $\delta_1 \in [0, 1]$ is arbitrary for the moment. We deduce that

$$|\text{Eq. (3.41)}| \leqslant (\tau\mu^{1+\gamma} + 2^{1-\delta_1} a^{\delta_1} \tau^{\delta_1} \mu + \delta_1 s_0)^{-\mu k \tau}. \tag{3.42}$$

When $k \geqslant k_0 \equiv (1+\gamma)(\tau\Lambda)^{-1}$ and $\mu \geqslant \Lambda$, then function $\mu \to \mu^{1+\gamma-\mu k\tau}$ is decreasing, therefore

$$\mu^{1+\gamma} e^{-\mu k\tau} \leqslant \Lambda^{1+\gamma} e^{-\Lambda k\tau}, \qquad k \geqslant (1+\gamma)(\tau\Delta)^{-1}. \tag{3.43}$$

For $k \leqslant k_0$, that function assumes its maximum at $\mu = (1+\gamma)(k\tau)^{-1}$ and we have

$$\mu^{1+\gamma} e^{-\mu k\tau} \leqslant ((1+\gamma)(\tau k)^{-1})^{1+\gamma} e^{-(1+\gamma)}, \qquad k \leqslant (1+\gamma)(\tau\Lambda)^{-1}. \tag{3.44}$$

It follows then from (3.43)–(3.44), that

$$\sum_{k=1}^{\infty} e^{\lambda k t} \operatorname*{Sup}_{\mu \geqslant \Lambda} \tau\mu^{1+\gamma} e^{-\mu k\tau} \leqslant K_7 \begin{cases} (1 + |\mathrm{Log}(\tau\Lambda)|), & \text{when} \quad \gamma = 0, \\ \tau^{-\gamma}, & \text{when} \quad \gamma > 0. \end{cases} \tag{3.45}$$

Concerning the second term in (3.42), we consider two cases. First, when $s_0 \leqslant 1$, we take $\delta_1 = 1 - \gamma$. The analogue of (3.45) reads

$$\sum_{k=1}^{\infty} e^{\lambda k\tau} \operatorname{Sup} \tau^{\delta_1} \mu^{\gamma + \delta_1 s_0} e^{-\mu k\tau} \leqslant K_8 \tau^{-\gamma} \tag{3.46}$$

with the only exception that if $\gamma = 0$ and $s_0 = 1$, we must replace the r.h.s. of (3.46) by $K_8(1 + |\mathrm{Log}\, \tau\Lambda|)$. Then, for $s_0 > 1$, we take $\delta_1 < (1-\gamma)/s_0$. The analogue of (3.45) reads this time

$$\sum_{k=1}^{\infty} e^{\lambda k\tau} \operatorname*{Sup}_{\mu \geqslant \Lambda} \tau^{\delta_1} \mu^{\gamma + \delta_1 s_0} e^{-\mu k\tau} \leqslant K_{10} \tau^{-1 + (1-\gamma)/s_0} (1 + \mathrm{Log}\, |\tau\Lambda|). \tag{3.47}$$

Returning to (3.36), we have to integrate with respect to σ the estimates (3.45), (3.46), and (3.47). Since these are independent of σ, we find

$$|\text{l.h.s. of Eq. (3.36)}| \leqslant \tau(|\text{Eq. (3.45)}| + \begin{cases} |\text{Eq. (3.46)}| & \text{for} \quad s_0 \leqslant 1, \\ |\text{Eq. (3.47)}| & \text{for} \quad s_0 > 1, \end{cases}$$

which is (3.36). ▮

Proof of Lemma 3.2. Here we consider

$$\mu^{\gamma} \left(e^{-(\mu + i\xi)k\tau} - (1+\tau\mu)^{-k} \left(\frac{1 - i\tau\xi/2}{1 + i\tau\xi/2} \right)^{+k} \right), \tag{3.48}$$

where $\mu \geqslant \Lambda$, $|\xi| \leqslant a\,|\mu|^{s_0}$ and $k \geqslant 1$. We write

$$\text{Eq. (3.48)} = \mu^{\gamma}(e^{-ik\tau\xi} - e^{-ik\theta})\, e^{-\mu k\tau} + \mu^{\gamma}(e^{-\mu k\tau} - (1+\tau\mu)^{-k})\, e^{-ik\theta},$$

where $\theta \equiv 2 \operatorname{Arctan}(\tau\xi/2)$, so that $(1 + i\tau\xi/2)/(1 - i\tau\xi/2) = \exp i\theta$. Here again we majorize separately the two terms. We have first that

$$\begin{aligned} |e^{ik\tau\xi} - e^{-ik\theta}| &= 2\,|\sin(k(\theta - \tau\xi)/2)| \\ &\leqslant 2^{1-\delta_2}\,|k|^{\delta_2}\,|\theta - \tau\xi|^{\delta_2}, \end{aligned}$$

where $\delta_2 \in [0, 1[$ is arbitrary for the moment. Then since for every $x \in \mathbb{R}$, $|\operatorname{Arctan} x - x| \leqslant x^2/2$, we deduce that

$$|e^{-ik\tau\xi} - e^{-ik\theta}| \leqslant 2^{1-2\delta_2} k^{\delta_2} \tau^{2\delta_2} |\xi|^{2\delta_2}, \qquad \forall \delta_2 \in [0, 1]. \tag{3.49}$$

Hence using that $|\xi| \leqslant a\mu^{s_0}$, we obtain

$$\begin{aligned} &|\mu^{\gamma}(e^{-ik\tau\xi} - e^{-ik\theta})\, e^{-\mu k\tau}| \\ &\quad \leqslant 2^{1-2\delta_2} a^{2\delta_2} \tau^{2\delta_2} \mu^{(\gamma + 2\delta_2 s_0)} k^{\delta_2}\, e^{-\mu k\tau}. \end{aligned} \tag{3.50}$$

The other term in (3.48) is bounded via a first order Taylor formula as follows

$$|\mu^{\gamma}(e^{-\mu k\tau} - (1+\tau\mu)^{-\kappa})| \leqslant \mu^{\gamma+2}\tau^2 k(1+\tau\mu)^{-k}/2. \tag{3.51}$$

For $k = 1$ and 2 it is in fact better to bound this term as follows

$$|\mu^{\gamma}(e^{-\mu k} - (1+\tau\mu)^{-k}| \leqslant \tau^{-\alpha}, \qquad k = 1 \text{ and } 2. \tag{3.52}$$

A careful study of the function of μ which appears in the right hand side of (3.51), shows that

$$\sum_{k=3}^{\infty} (\operatorname{Sup} |\mu^{\gamma}(e^{-\mu k\tau} - (1+\tau\mu)^{-k})|)\, e^{\lambda k} \leqslant K_{11} \tau^{-\alpha}. \tag{3.53}$$

Concerning (3.50), we write

$$\begin{aligned} S &= \sum_{1}^{\infty} (\operatorname*{Sup}_{\mu \geqslant \Lambda} |\mu^{\gamma}(e^{-ik\tau\xi} - e^{-ik\theta})\, e^{-\mu k\tau}|)\, e^{\lambda k\tau} \\ &\leqslant 2^{1-\delta_2} a^{2\delta_2} \tau^{2\delta_2} \left(\sum_{k \leqslant (g + 2\delta_2 S_0)/\tau\Lambda} k^{\delta_2} \left(\frac{k\tau}{\gamma + 2\delta_2 s_0} \right)^{-(\gamma + 2\delta_2 s_0)} e^{(\lambda k\tau)}\, e^{-(\gamma + 2\delta_2 s_0)} \right) \\ &\quad + \sum_{k > (\gamma + 2\delta_2 S_0)/\tau\Lambda} k^{\delta_2} \Lambda^{\gamma + \delta_2 s_0}\, e^{(\lambda - \Lambda)k\tau} \\ &= 2^{1-\delta_2} a^{2\delta_2} \tau^{2\delta_2} \{S_1 + S_2\}. \end{aligned} \tag{3.54}$$

Concerning S_2, we remark that for $k > (\gamma + 2\delta_2 S_0)/\tau\Lambda$ the function $x \to x^{\delta_2} e^{(\bar{\lambda} - \Lambda)x\tau}$ is decreasing and then

$$S_2 \leqslant \Lambda^{\gamma + 2\delta_2 s_0} \int_0^{+\infty} x^{\delta_2} e^{(\bar{\lambda} - \Lambda)x\tau}\, dx$$
$$= K_{12} \Lambda^{\gamma + 2\delta_2 s_0}((\bar{\lambda} - \Lambda)\tau)^{-(\delta_2 + 1)}.$$

Concerning S_1 we see that if $\delta_2(1 - 2S_0) - \gamma > -1$,

$$S_1 = K_{13} \tau^{-(\gamma + 2\delta_2 s_0)} \sum_{k \leqslant (\gamma + 2\delta_2 S_0)/\tau\Lambda} k^{\delta_2(1 - 2s_0) - \gamma}$$
$$\leqslant K_{14} \tau^{-(\gamma + 2\delta_2 s_0)} (\tau^{-1})^{(\delta_2(1 - 2s_0) - \gamma + 1)}$$
$$= K_{14} \tau^{-(\gamma + 2\delta_2 s_0) + \delta_2(1 - 2s_0) - \gamma + 1}$$
$$= K_{14} \tau^{-\delta_2 - 1}$$

if $\delta_2(1 - 2s_0) - \gamma = -1$ we find that

$$S_1 \leqslant K_{13} \tau^{-\gamma + 2\delta_2 s_0)} \operatorname{Log} |\tau\Lambda|.$$

Suppose first that $s_0 < 1$ and choose $\delta_2 = 1 - \gamma$; we have $\delta_2(1 - 2s_0) - \gamma < 1$ and then

$$S \leqslant 2^{1 - \delta_2} a^{2\delta_2} \tau^{-\gamma} (K'_{12} + K_{14}).$$

If $s_0 \geqslant 1$ we choose $\delta_2 = (1 - \gamma)/(2S_0 - 1)$ and find

$$S_1 \leqslant \tau^{(1 - \gamma)/(2s_0 - 1) - 1}.$$

Therefore, returning to (3.48), and using (3.52), (3.53), and (3.54) we find for the left hand side of (3.37)

$$\begin{aligned} \text{l.h.s. of (3.37)} &\leqslant K_{15} \tau^{1 - \gamma} && \text{if} \quad s_0 < 1 \text{ and } \gamma \in \,]0, 1[\\ &\leqslant K_{15} \tau (1 + |\log(\tau\Lambda)|) && \text{if} \quad s_0 < 1 \text{ and } \gamma = 0 \\ &\leqslant K_{15} \tau^{(1 - \gamma)/(2s_0 - 1)} (1 + |\log(\tau\Lambda)|) && \text{if} \quad s_0 \geqslant 1. \end{aligned}$$

3.3. *The Proof of Theorem* 3.1

This proof is now straightforward using Proposition 3.2. Indeed, by (3.0) and Theorem 1.1, we have $\phi = \mathscr{T}\phi$, $\phi \in \mathscr{F}_{1/4}$. On the other hand for $\tau\Lambda \leqslant 1$, we have by Theorem 2.2, $\phi_\tau = \mathscr{T}_\tau \phi_\tau$, $\phi_\tau \in \mathscr{F}_{1/4}$. Hence we can write

$$\phi - \phi_\tau = \mathscr{T}\phi - \mathscr{T}_\tau \phi_\tau$$
$$= \tau\phi - \mathscr{T}_\tau \phi + \mathscr{T}_\tau \phi - \mathscr{T}_\tau \phi_\tau.$$

Now, according to (2.21) ($\kappa = 7/10$ since $l = \frac{1}{4}$)

$$\|\mathscr{T}_\tau \phi - \mathscr{T}_\tau \phi_\tau\|_\alpha \leqslant (7/10)\, \|\phi - \phi_\tau\|_\alpha.$$

It follows that

$$\begin{aligned}\|\phi - \phi_\tau\|_\alpha &\leqslant (10/3)\, \|\mathscr{T}\phi - \mathscr{T}_\tau \phi\|_\alpha \\ &\leqslant (\text{by Eq. (3.20)}) \\ &\leqslant (10C_0/3)\, \tau^\zeta (1 + |\mathrm{Log}\, \tau A|)^\varepsilon\end{aligned}$$

which is exactly the desired result. ∎

4. Applications

In this section we develop two types of applications. The first one concerns space–inhomogeneous complex amplitude equations (Ginzburg–Landau like equations). The second one is related to the Korteweg–deVries–Burgers equation. In both cases, the linear part is not self-adjoint (i.e., $C \neq 0$), and this has partly motivated this work. We also note that many other applications are possible, in particular reaction-diffusion equations, Kuramoto–Sivashinsky equations, Cahn–Hillard equation, ... For these equations we have $C = 0$, i.e., the linear part is self-adjoint, existence of inertial manifolds in the continuuous case has been proved in [11, 9, 20, 21, and 16] depending on the equation considered. Here also, our results provide existence and convergence of inertial manifold for the time-discretized equation. Finally, we refer to [5] conerning other applications.

4.1. *Complex Amplitude Equations*

In this part we consider the following family of equations.

$$\frac{\partial u}{\partial t} - (1 + i\alpha)\, \Delta u + (1 + i\beta)\, f(|u|^2) u = ru, \tag{4.1}$$

where α, β, r are real parameters, $u = u(x, t)$ is a complex valued function defined on $\Omega \times \mathbb{R}_+$, Ω a bounded open set in $\mathbb{R}^d$, $d = 1$ or 2. Two typical cases are $f(s) = s$ and $f(s) = \pm s(1 + \delta s)^{-1}$, $\delta > 0$. In the former case, (4.1) is the well-known Ginzburg–Landau pde which describes the amplitude evolution of instability waves in a very large variety of dissipative systems in fluid mechanics. In particular systems which are close to criticality, e.g., in Taylor–Couette flow, Bénard convection or plane Poiseulle flow .... In the cases $f(s) = \pm s(1 + \delta s)^{-1}$, $\delta > 0$, (4.1) is a Laser equation [6] occurring in the study of propagation of intensive light in a nonlinear medium cavity. Other cases can arise, for example, $f(s) = -s + s^2$.

Various boundary conditions can supplement (4.1): Dirichlet BC

$$u(x, t) = 0, \qquad x \in \partial\Omega, t \geqslant 0, \tag{4.2$_D$}$$

where $\partial\Omega$ denotes the boundary of Ω, or Neumann BC

$$\frac{\partial u}{\partial v}(x, t) = 0, \qquad x \in \partial\Omega, t \geqslant 0, \tag{4.2$_N$}$$

where v denotes the outward unit normal on $\partial\Omega$, or finally periodic BC: $\Omega =]0, L_i[^d$,

$$u(\cdot, t) \text{ is } \Omega\text{-periodic}. \tag{4.21$_p$}$$

With regards to the applications above, we shall make on the function $f \in \mathscr{C}^2(\mathbb{R}_+, \mathbb{R})$ one of the two hypotheses either:

$$\begin{gathered} \exists \sigma > 0, C_1 \geqslant 0, R \geqslant 0, \underline{f} > 0, \bar{f} > 0 \text{ such that} \\ f(s) \geqslant \underline{f} s^\sigma - C_1, |f'(s)| \leqslant \bar{f}(s)| \leqslant \bar{f} s^{\sigma-1}, \forall s \geqslant R, \end{gathered} \tag{4.3}$$

or

$$\exists \omega \geqslant 0, \qquad z_1 f(|z_1|^2) - z_2 f(|z_2|^2)| \leqslant \omega\, |z_1 - z_2|, \qquad \forall z_i \in \mathbb{C}. \tag{4.4}$$

We note that the case $f(s) = s$ and $f(s) = -s + s^2$ satisfies (4.3), while $f(s) = \pm s(1 + \delta s)^{-1}$ satisfies (4.4).

4.1a. *The Functional Setting.* We set $H = L^2(\Omega)^2$, where $L^2(\Omega)$ is the usual Lebesgue space of order 2 (of real valued function). We also denote by $H^m(\Omega)$ the Sobolev space

$$H^m(\Omega) = \{v \in L^2(\Omega), D^\alpha v \in L^2(\Omega), \forall \alpha, |\alpha| \leqslant m\}.$$

Complex functions $v = v_1 + iv_2$, $v_i \in \mathbb{R}$, are identified with pairs $\{v_1, v_2\} \in \mathbb{R}^2$ and the scalar product on H is then

$$(v, w) = \operatorname{Re} \int_\Omega v\bar{w}\, dx = \int_\Omega (v_1 w_1 + v_2 w_2)\, dx. \tag{4.5}$$

The bounded operator A is defined as

$$Av = -\Delta v + v, \qquad v \in D(A), \tag{4.6}$$

where

$$D(A)=\begin{cases}\{v\in H^2(\Omega),\ v=0 \text{ on } \partial\Omega\}, \text{ or}\\ \{v\in H^2(\Omega),\ \partial v/\partial \nu=0 \text{ on } \partial\Omega\}, \text{ or}\\ \{v\in H^2(\Omega),\ v \text{ and } \partial v/\partial \nu \text{ are } \Omega\text{-periodic}\},\end{cases}$$

according to the boundary condition (4.2). The operator C is unbounded too,

$$C=i\alpha(A-I) \tag{4.7}$$

it commutes with A, is squew-symmetric and continuous from $D(A^{s+1})$ into $D(A^s)$, $\forall s\in\mathbb{R}$, i.e., we have $s_0=1$. With these notations, (4.1) enters in the abstract form (1.4) provided we set

$$F(v)=(1+r)v+(1+i\beta)\,f(|v|^2)v. \tag{4.8}$$

4.1b. *Lipschitz Properties of F.* It is clear that in the case $f(s)=s$, we cannot expect F to be a globally Lipschitz mapping on some $D(A^\alpha)$, $\alpha\in\mathbb{R}$. We shall return to this later, and first consider the case (4.4). Indeed in that case,

$$\int_\Omega |F(v)-F(w)|^2\,dx\leqslant(|1+r|+(1+\beta^2)^{1/2}\omega)\int_\Omega |v-w|^2\,dx$$

and (1.6) and (1.7) hold true with $\alpha=\tau=0$, $L_F=(|1+r|+(1+\beta^2)^{1/2}\omega)$.

Let us now consider the case (4.3). Local in time existence and uniqueness of solutions to (4.1)–(4.2) is very classical. Given $u_0\in H$, we denote by $S(t)\,u_0=u(t)$ the maximal solution

$$u\in\mathscr{C}([0,T[,L^2(\Omega)^2)\cap\mathscr{C}(]0,T[;H^1(\Omega)^2),\qquad T=T_{\max}(u_0),$$

satisfying $u(0)=u_0$. As we are going to see, a priori estimates on u will prove that $T_{\max}(u_0)=+\infty$, $\forall u_0\in H$. In fact $S(t)$ is bounded on H:

PROPOSITION 4.1. *We assume that f satisfies* (4.3) *with $\sigma\leqslant 2$ when $d=1$ and $\sigma\leqslant 1$ when $d=2$. There exists $\rho\geqslant 0$ depending only on the data α, β, r, Ω and f such that for every $u_0\in H$, $|u_0|\leqslant R$ there exists $t_0(R)$ for which the solution to* (4.1)–(4.2) *with initial data u_0 satisfies*

$$\|u\|_1^2\equiv\int_\Omega(|u(x,t)|^2+|\nabla u(x,t)|^2)\,dx\leqslant\rho^2,\qquad\forall t\geqslant t_0(R). \tag{4.9}$$

This shows that the semigroup $S(t)$ possesses a bounded absorbing set $B_\rho = \{v \in H^1(\Omega)^2,\ \|v\|_1 \leqslant \rho\}$.[1] Before giving the sketch of the proof of this result, which is postponed to the end of this section we discuss the applicability of the abstract framework of Section 1 to (4.1)–(4.2).

(i) The one dimensional case. In that case since the norm in $H^1(\Omega)$ bound the sup-norm, we deduce from Theorem 4.1 that $S(t)$ possesses a bounded absorbing set B_∞ in $L^\infty(\Omega)$: $\exists \rho_\infty > 0$ such that $B_\infty = \{v \in L^\infty(\Omega),\ \mathrm{Sup}_{x \in \Omega} |v(x)| \leqslant \rho_\infty\}$ is absorbing see also Remark 4.1. Since we are concerned here with the long time behavior of solutions to (4.1)–(4.2), we replace F in (4.8) by

$$F(v) = (1 + r)v + (1 + i\beta)\, \theta(|v|^2 \rho_\infty^{-2})\, f(|c|^2)v, \tag{4.10}$$

where θ is a cut-off function: $\theta \in C^\infty(\mathbb{R}_+)$ and $\theta(s) = 1$ for $s \in [0, 2]$ and $\theta(s) = 0$ for $s \geqslant 3$. It is clear that after a transient period, a solution of (4.1)–(4.2) is a solution to (1.4) with F given in (4.10). This allows us to study inertial manifolds for this last equation. And this is relevant as far as the long time behavior of the solution to (4.1)–(4.2) is concerned. Now it is clear that the "new" f, i.e., $\theta(s\rho_\infty^{-1}) f(s)$ satisfies (4.4) and we conclude as before that we can take $\alpha = \gamma = 0$ in (1.6)–(1.7).

(ii) The two dimensional case. In that case we take instead of (4.8),

$$F(v) = (1 + r)v + (1 + i\beta)\, \theta(\|v\|_1^2 \rho^{-2})\, f(|v|^2)v, \tag{4.11}$$

where θ is as before and ρ was given in Theorem 4.1. We claim that there exists a constant ω_1, which only depends on r, β, ρ, Ω, and f such that

$$\|F(v) - F(w)\|_1 \leqslant \omega_1 \|v - w\|_1, \qquad \forall v, w \in H^1(\Omega)^2. \tag{4.12}$$

Let us note that according to (4.12), we see that (1.6)–(1.7) hold true with $\alpha = \frac{1}{2}$, $\gamma = 0$, L_F/ω_1.

Concerning the proof of (4.12), we note first that the linear part in (4.11) is harmless. Hence we have to show that the nonlinear part satisfies (4.12). Since this part is compactly supported in $H^1(\Omega)^2$, we only have to check that $v \to \theta(\|v_1\|^2 \rho^{-2}) f(|v|^2)$ is a locally Lipschitz function from $H^1(\Omega)$ into itself.

But now this follows by the fact that we are in the case where $\Omega \subset \mathbb{R}^2$ and $H^1(\Omega)$ is continuously imbedded in $L^p(\Omega)$, $\forall p < \infty$.

Remark 4.1. In the case where $\alpha = 0$, (4.2) is a special kind of reaction diffusion system of two equations for which maximum principle is available

[1] A set B_α is said to be absorbing for $S(t)$ if for every bounded set B, there exists $T(B)$ s.t. $S(t)\, B \subset B_\alpha$ for $t \geqslant T(B)$.

(see [7] concerning the case where $f(s)=s$ and [5] for more general cases). In that case, one can show [5] the existence of a bounded absorbing set in $L^\infty(\Omega)$ without any restriction on the space dimension. Hence one can replace F in (4.8) by (4.10) and obtain (1.6)–(1.7) with $\alpha=\gamma=0$.

4.1c. *Applications of the Results in the One Dimensional Case.* In the one dimensional case, we set $\Omega =]0, L[$, $L>0$. The eigenelements of the operator A are well known. In the cases (4.1) and (4.2) the eigenvalues $\lambda_k = \Lambda_k$, $k \geqslant 1$, are distincts and we have $\Lambda_k = 1+(k-1)^2\Pi^2L^2$. In the case (4.3), we have $\Lambda_k = 1+4(k-1)^2\Pi^2L^{-2}$ with multiplicity $m_k=2$ except for m_1 which is equal to 1. The condition $\Lambda_{N+1} \geqslant 3L_F^2\Lambda_1^{2\gamma-1}/2$, reads (here $\gamma=0$, $\Lambda_1=1$)

$$N \geqslant C_0 L_F L,$$

where C_0 is an absolute constant. The more drastic condition $\Lambda_{N+1} - \Lambda_N \geqslant 30L_F(\Lambda_N^\gamma + \Lambda_{N+1}^\gamma)$, reads

$$N \geqslant C_1 L_F L^2, \qquad L \geqslant 1, \tag{4.13}$$

where C_1 is an absolute constant. This last constraint on N is more stringent than the former since the relevant case to consider is $L \gg 1$.

The fractional step scheme proposed in Theorem 2.2, reads in case of (4.1),

$$\begin{gathered}(u^{n+1/2}-u^n)/\tau - \Delta u^{n+1/2} + (1+i\beta)\,f(|u^n|^2)u^n = ru^n,\\ (u^{n+1}-u^{n+1/2})/\tau - i\alpha\,\Delta(u^{n+1}+u^{n+1/2})/2 = 0,\end{gathered} \tag{4.14}$$

which are supplemented with one of the BC (4.2). Let us first deal with the case, where f satisfies (4.4). In that case $L_F = |1+r| + (1+\beta^2)^{1/2}\omega$, and summarizing Theorems 1.1, 2.2, and 3.1 we deduce

THEOREM 4.1. *We assume that f satisfies* (4.4). *Equations* (4.1)–(4.2) *on* $\Omega =]0, L[$, $L>0$ *possesses an M-dimensional manifold in* $L^2(\Omega)^2$, $M=M_\phi$, *where*

$$M \leqslant C_2(|1+r| + (1+\beta^2)^{1/2}\omega)L^2 \tag{4.15}$$

and C_2 *is a universal constant. For every* $\tau>0$, *satisfying* $\tau L^2(|1+r| + (1+\beta^2)^{1/2}\omega)^2 \leqslant C_3$, C_3 *a universal constant, the discrete iteration* (4.14) *possesses also an M-dimensional inertial manifold in* $L^2(\Omega)^2$, $M_\tau = M_{\phi_\tau}$. *Moreover we have the error estimate*

$$\|\phi-\phi_\tau\|_0 \leqslant K\tau(1+|\mathrm{Log}\,\tau|).$$

Comments. (1) This result provides an inertial manifold for (4.1)–(4.2) without changing the nonlinear terms. To our knowledge this is the first result of existence of an inertial manifold for the original partial differential equation.

(2) The scheme (4.14) can be actually implemented by using collocation-spectral methods.

(3) On the inertial manifold, (4.14) becomes a finite dimensional iteration on $\mathbb{R}^M$, which represents the long time behavior of the original problem (4.1)–(4.2).

(4) In the case where $f(s) = -s(1+\delta s)^{-1}$, and $r \geqslant 0$, (4.1)–$(4.2)_{N \text{ or } P}$ possesses solutions which grow exponentially as $t \to +\infty$, for example those which are independent of x. Hence (4.1)–$(4.2)_{N \text{ or } P}$ are not dissipative dynamical systems. However, our results apply, and they show in particular that these equations have a *finite diensional behavior*. This is the first result in that direction for a nondissipative infinite dimensional dynamical system.

If f satisfies (4.3), we take F given by (4.11), i.e., we consider instead of (4.1)

$$\frac{\partial u}{\partial t} - (1+i\alpha)\,\Delta u + (1+i\beta)\,\theta(\|u\|_1^2\,\rho^{-2})\,f(|u|^2)u = ru. \tag{4.1$'$}$$

In that case Theorem 4.1 has an analogue for (4.1)′–(4.2), provided we replace ω by ω_1 (see (4.12)); the inertial manifolds are obtained in $H^1(\Omega)^2$ and the error estimate is now with respect to the norma of that space. Let us also mention that the case $f(s)=s$, i.e., the case of the Ginzburg–Landau equation, is addressed in [23]. An inertial manifold for (4.1) is constructed in that case.

4.1.d. *The Two Dimensional Case.* In that case, as is well known, $\lambda_k = (4\Pi k/\text{vol}\,\Omega) + o(k)$, where $\text{vol}\,\Omega$ denotes the area of Ω. This asymptotic expansion does not allow us to find arbitrarily large gaps in the spectrum of A, in order to satisfy $\Lambda_{N+1} - \Lambda_N \geqslant 30 L_F(\Lambda_N^\gamma + \Lambda_{N+1}^\gamma)$ (here $\gamma = 0$).

It is clear that the condition $\Lambda_{N+1} \geqslant 3L_F^2/2\Lambda_1^{2\gamma-1}$ is satisfied as soon as N is large. Hence in order to show the spectral gap condition $\Lambda_{N+1} - \Lambda_N \geqslant 60 L_F$, for arbitrary L_F, we must have

$$\limsup_{N \to +\infty} (\Lambda_{N+1} - \Lambda_N) = +\infty. \tag{4.16}$$

One can answer positively in the case where $\Omega =]0, L_1[x]0, L_2[$ is a rectangle and $(L_1/L_2)^2$ is rational [15]. Indeed, in that case the eigenvalues of A are known, they read $1+\Pi^2(k_1^2L_1^{-2}+k_2^2L_2^{-2})$, where k_1, k_2 are integers. A result from number theory (Richards [18]) then implies (4.25). An analogue of Theorem 4.2 then holds.

4.1e. *Sketch of the Proof of Proposition* 4.1. Let us consider a msooth solution u to Eqs. (4.1)–(4.2). Multiplying by $\bar{u}$ (resp. $-\Delta\bar{u}$) and integrating on Ω the real part of the resulting identities we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx+\int_\Omega |\nabla u|^2\,dx+\int_\Omega f(|u|^2)\,|u|^2\,dx$$

$$=r\int_\Omega |u|^2\,dx, \tag{4.17}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx+\int_\Omega |\Delta u|^2\,dx+\operatorname{Re}(1+i\beta)\int_\Omega \nabla(f(|u|^2)u)\,\nabla\bar{u}\,dx$$

$$=r\int_\Omega |\nabla u|^2\,dx. \tag{4.18}$$

According to (4.3), there exists C_1 such that $f(|u|^2)\geqslant \underline{f}|u|^{2\sigma}-C_1$. Now there exists C_2 such that $\underline{f}s^{2\sigma+2}-(C_1+1+r)s^2\geqslant \underline{f}s^{2\sigma+2}/2-C_2$, and we deduce from (4.13)

$$\frac{d}{dt}\int_\alpha |u|^2\,dx+\int_\Omega |u|^2\,dx+2\int_\Omega |\nabla u|^2\,dx+\underline{f}\int_\Omega |u|^{2\sigma+2}\,dx\leqslant C_3. \tag{4.19}$$

This implies

$$\int_\Omega |u(x,t)|^2\,dx\leqslant\left(\int_\Omega |u_0(x)|^2\,dx\right)e^{-t}+C_3(1-e^{-t}),\qquad \forall t\geqslant 0. \tag{4.20}$$

Concerning (4.18), we have

$$\operatorname{Re}(1+i\beta)\int_\Omega (\nabla f(|u|^2)u)\,\nabla\bar{u}\,dx$$

$$=\int_\Omega f(|u|^2)\,|\nabla u|^2\,dx+\operatorname{Re}(1+i\beta)\int_\Omega f'(|u|^2)\,u\,\nabla\bar{u}\operatorname{Re}(u\,\nabla\bar{u})\,dx$$

and since $f(|u|^2)\geqslant -C_1$, we conclude with (4.3) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx+\int_\Omega |\Delta u|^2\,dx$$

$$\leqslant (r+C_1)\int |\nabla u|^2\,dx+C_4\int |u|^{2\sigma}\,|\nabla u|^2\,dx. \tag{4.21}$$

We bound the integral $\int |u|^{2\sigma} |\nabla u|^2 \, dx$ by using the Holder inequality

$$\int |u|^{2\sigma} |\nabla u|^2 \, dx \leqslant \left(\int |u|^{2\sigma+2} \, dx \right)^{\sigma(\sigma+1)} \left(\int |\nabla u|^{2(\sigma+1)} \, dx \right)^{1(\sigma+1)} \tag{4.22}$$

Then due to the following Gagliardo–Nirenberg inequality [17] on two dimensional domains

$$\left(\int_\Omega |\nabla u|^{2(\sigma+1)} \, dx \right) \leqslant C_\sigma \left(\int_\Omega |\nabla u|^2 \, dx \right) \left(\int_\Omega |\Delta u|^2 \, dx \right)^\sigma,$$

where C_σ depends only on σ and Ω, we find in (4.18)

$$\int |u|^{2\sigma} |\nabla u|^2 \, dx \leqslant C_5 \left(\int |u|^{2\sigma+2} \, dx \right)^{\sigma/(\sigma+1)} \left(\int |\nabla u|^2 \, dx \right)^{\sigma/(\sigma+1)}$$

(by the Young inequality)

$$\leqslant \frac{1}{2C_4} \int_\Omega |\Delta u|^2 \, dx + C_6 \left(\int_\Omega |\nabla u|^2 \, dx \right) \left(\int_\Omega |u|^{2\sigma+2} \, dx \right)^\sigma. \tag{4.23}$$

Reporting this estimate in (4.17) then leads to

$$\frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\Delta u|^2 \, dx$$
$$\leqslant 2 \left(r + C_1 + C_6 \left(\int_\Omega |u|^{2\sigma+2} \, dx \right) \right)^\sigma \int_\Omega |\nabla u|^2 \, dx. \tag{4.24}$$

We first claim that (4.15), (4.16), and (4.20) prove that local in time solutions to (4.1)–(4.2) are in fact global. Indeed thanks to (4.16), integrating (4.15) between 0 and T shows that

$$\int_0^T \int_\Omega \left(|u|^{2\sigma+2} + |\nabla u|^2 \right) dx \leqslant C_7(T) < \infty. \tag{4.25}$$

In the two dimensional case, we have assumed that $\sigma \leqslant 1$, therefore returning to (4.24) and using (4.25) and Gronwall's Lemma it follows that for every $T > 0$,

$$\operatorname*{Sup}_{0 \leqslant t \leqslant T} t \int_\Omega |\nabla u(x, t)|^2 \, dx \leqslant C_8(T) < \infty.$$

And now global in time existence follows by classical means. So far we have not proved a time uniform bound on $|u(t)|_1$. It is necessary to make use of a time uniform Gronwall's Lemma and we refer to [5] concerning the details of the proof of (4.9), which indeed follows from (4.19), (4.20), and (4.21).

In the one dimensional case, the relevant Gagliardo–Nirenberg inequality is

$$\left(\int_\Omega |\Delta u|^{2(\sigma+1)}\,dx\right) \leqslant C_\sigma \left(\int_\Omega |\nabla u|^2\,dx\right)^{1+(\sigma/2)} \left(\int_\Omega |\Delta u|^2\,dx\right)^{\sigma/2}.$$

And this leads, instead of (4.24), to

$$\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\Delta u|^2\,dx$$
$$\leqslant 2\left(r + C_1 + C_7\left(\int_\Omega |u|^{2\sigma+2}\right)^{2\sigma/(\sigma+2)} \int_\Omega |\nabla u|^2\,dx\right).$$

In that case $\sigma \leqslant 2$ so that $2\sigma/(\sigma+2) \leqslant 1$ and we argue as in the previous case.

4.2. *Pseudo Korteweg–de Vries–Burgers Equations*

Here we consider the following family of equations

$$\partial_t u + \partial_x^{2p+1} u + \kappa u\,\partial_x u + (-\partial_x^2)^s u = f, \tag{4.26}$$

where the unknown function $u = u(x, t)$ is real and 2Π-space periodic

$$u(x + 2\Pi, t) = u(x, t), \qquad \forall x \in \mathbb{R}, \forall t \geqslant 0. \tag{4.27}$$

The function f and the constant κ are given, p is an integer (usually $p = 1$ or 2) and $(-\partial_x^2)^s$ is the pseudo differential operator with symbol $|k|^{2s}$.

4.2a. *The Functional Setting.* The Fourier coefficients of a 2Π-periodic function $v = v(x)$ are denoted by v_k, $k \in \mathbb{Z}$, i.e., $v(x) = (2\Pi)^{-1/2} \sum_{k\in\mathbb{Z}} v_k$ exp ikx. Given $\sigma \leqslant 0$, we denote by H_L^σ the usual fractional Sobolev space of periodic functions:

$$H_L^\sigma = \left\{ v \in L^2(0, L),\ \sum_{k\in\mathbb{Z}} (1 + k^2)^\sigma\,|v_k|^2 < \infty \right\}.$$

We denote by $H_L^0 = L_L^2 = L^2(0, L)$ and by $\mathring{H}_L^\sigma = \{v \in H_L^\sigma, v_0 = 0\}$ the subspace of function in H_L^σ with vanishing mean. With regards to the setting of Section 1, we take $H = \mathring{L}^2(0, L)$ and set

$$(Av)_k = |k|^{2s} v_k, \qquad (Cv)_k = (-1)^p i k^{2p+1} v_k, \qquad D(A) = \mathring{H}_L^{2s}.$$

It is clear that C is squew-symmetric and maps continuously $D(A^{s+s_0})$ into $D(A^{s_0})$, $s_0=(2p+1)/(2s)$. Assuming that $f\in \mathring{L}^2_L$, the function F defined by $F(v)=\kappa v\,\partial_x v-f$ is locally lipschitzian from H^1_L into L^2_L. Indeed, assuming that $v\in H^1_L=D(A^{1/2s})$ we write $F(v)=\partial_x(\kappa v^2/2)-f$, since H^1_L is an algebra, $\kappa v^2/2\in H^1_L$ and then $F(v)\in L^2_L$. Moreover, since

$$F(v)-F(w)=\kappa\partial_x(v^2-w^2)/2$$

we have ($|\,|\sigma|_\sigma$ is the norm $|A^\sigma\cdot|$)

$$|F(v)-F(w)|_0\leqslant\frac{k}{2}\,|v^2-w^2|_0\leqslant\frac{k}{2}\,|v+w|_{L^\infty}\,|v-w|_0.$$

The sup-norm $|w|_{L^\infty}$ of w is bounded by $|\,|_{1/2s}$ which is the H^1_L-norm, therefore this last estimate shows that F is *locally* lipschitzian from $D(A^\alpha)$ into $D(A^{\alpha-\gamma})$, $\alpha=1/2s$. It is clear that this F is not globally lipschitzian. We are going to use here, like in the previous case, the existence of a bounded absorbing set and a truncation method.

4.2b. *Time Uniform Estimates.* We assume that $u_0\in L^2_L$ and $f\in L^2_L$, it is classical that (4.26)–(4.27) possesses a unique solution satisfying

$$u\in\mathscr{C}(\mathbb{R}_+;L^2_L)\cap L^2(0,T;H^s_L),\qquad\forall T>0,$$

and $u(0)=u_0$. Denoting by $S(t)$ the semigroup $\mathring{L}^2_L\ni u_0\to S(t)u_0=u(t)$, we have

PROPOSITION 4.2. *The bounded sets* B_ε,

$$B_\varepsilon=\{v\in H^1_L,\ |v|_0^2\leqslant(1+\varepsilon)\,|f|_0^2,\ |v_x|_0^2\leqslant C_\varepsilon(f)\},$$

where $C_\varepsilon(f)=|f|_0^2+C_0(1+\varepsilon)^4\kappa^8\,|f|_0^{10}$ *and* C_0 *is a numerical constant, are positively invariant by* $S(t)$:

$$S(t)\,B_\varepsilon\subset B_\varepsilon,\qquad\forall\varepsilon\geqslant 0,\forall t\geqslant 0.$$

Moreover for every $\varepsilon>0$, B_ε *is a bounded absorbing set for* $S(t)$ *in* H^1_L, *i.e., for every bounded set* B *in* H^1_L, *there exists* $T_\varepsilon(B)$ *such that*

$$S(t)B\subset B_\varepsilon,\qquad\forall t\geqslant T_\varepsilon(B),\forall\varepsilon>0.$$

Remark 4.2. (i) In general, the B_ε are not balls in H^1_L. (ii) This result holds true for $s\geqslant 1$, and $T_\varepsilon(B)$ can be chosen independent of $s\geqslant 1$.

Proof. According to (4.26)–(4.27), we have

$$\frac{1}{2}\frac{d}{dt}\int u^2\,dx+\int u(-\partial_x^2)^s\,u\,dx=\int uf\,dx,$$

$$\frac{1}{2}\frac{d}{dt}\int(\partial_x u)^2\,dx+\int(-{}_x^4u)(-\partial_x^2)^s u\,dx$$

$$=\kappa\int u(\partial_x u)(\partial_{xx}u)\,\partial x+\int(-\partial_x^2 u)\,f\,dx,$$

where the bound of integration, when they are omitted, are $x=0$, $x=L$. It follows that

$$\frac{1}{2}\frac{d}{dt}|u|_0^2+|u_x|_0^2\leqslant\int uf\,dx, \tag{4.28}$$

$$\frac{1}{2}\frac{d}{dt}|u_x|_0^2+|u_{xx}|_0^2\leqslant\kappa\int uu_xu_{xx}\,dx-\int u_{xx}f\,dx. \tag{4.29}$$

Since $|u_x|_0\geqslant|u|_0$, we deduce from (4.28)

$$|u(t)|_0\leqslant|u_0|\,e^{-t}+|f|_0(1-e^{-t}). \tag{4.30}$$

Then we recall that

$$M_{L^\infty}^2=\operatorname*{Sup}_{0\leqslant x\leqslant L}|v(x)|^2\leqslant 2\,|v|_0\,|v_x|_0,\qquad\forall v\in H_L^1, \tag{4.31}$$

and estimate thanks to this the cubic term in (4.29). Indeed

$$\int uu_xu_{xx}\,dx=\frac{1}{2}\int u(u_x^2)_x=-\frac{1}{2}\int u_x^3\leqslant|u_x|_{L^\infty}\,|u_x|_0^2/2$$

$$\leqslant 2^{-1/2}\,|u_x|_0^{5/2}\,|u_{xx}|_0^{1/2}.$$

Now since $|u_x|_0^2\leqslant|u|_0\,|u_{xx}|_0$, we conclude by using the Young inequality, $8ab\leqslant 7a^{8/7}+b^8$ that

$$\kappa\int uu_xu_{xx}\,dx\leqslant\frac{1}{4}\,|u_{xx}|_0^2+\frac{C_0\kappa^8}{2}\,|u|^{10}, \tag{4.32}$$

where C_0 is a numerical constant $(7^7 2^{-13})$. Returning to (4.29) and using (4.32) and $|\int u_{xx}f\,dx|\leqslant|f|_0\,|u_{00}|_0\leqslant|f|_0^2+|u_{xx}|_0^2/4$, $|u_x|_0\leqslant|u_{xx}|_0$, we finally find

$$\frac{d}{dt}|u_x|_0^2+|u_x|_0^2\leqslant|f|_0^2+c_0\kappa^8\,|u|^{10}. \tag{4.33}$$

We are now in a position to show that the B_ε are positively invariant by $S(t)$. We take $u_0 \in B_\varepsilon$, $\varepsilon \geqslant 0$. According to (4.30), we see that $|u(t)|_0^2 \leqslant (1+\varepsilon)\,|f|_0^2$, $\forall t \geqslant 0$. Then by (4.33),

$$\frac{d}{dt}\,|u_x|_0^2 + |u_x|_0^2 \leqslant |f|_0^2 + C_0 K^8 (1+\varepsilon)^5\,|f|_0^5 = C_\varepsilon(f).$$

Since $|u_{0x}|_0^2 \leqslant C_\varepsilon(f)$, we deduce that $|u_x(t)|_0^2 \leqslant C_\varepsilon(f)$, $\forall t \geqslant 0$ and the invariance of B_ε is proved.

Finally, for $\varepsilon > 0$, we consider a bounded set B in H_L^1, $B \subset \{v \in H_L^1,\ |v|_0^2 + |v_x|_0^2 \leqslant R^2\}$. According to (4.30), for $u_0 \in B$ we have

$$|u(t)|_0 \leqslant (1+\varepsilon)\,|f|_0^2 \qquad \text{for} \quad t \geqslant t_\varepsilon(R) = \operatorname{Log}(\varepsilon\,|f|_0^2/R). \tag{4.34}$$

On the other hand, thanks to (4.30) and (4.33),

$$\frac{d}{dt}\,|u_x|_0^2 + |u_x|_0^2 \leqslant |f|_0^2 + C_0 K^8 (|u_0|^{10}\,e^{-t} + |f|_0^{10}(1-e^{-t})),$$

which gives after integration

$$|u_x(t)|_0^2 \leqslant (R^2 + C_0 \kappa^8 R^{10} t)\,e^{-t} + |f|_0^2 + C_0 K^8\,|f|_0^{10}, \qquad \forall t \geqslant 0. \tag{4.35}$$

It is clear that there exists $T_\varepsilon(R) \geqslant t_\varepsilon(R)$ such that the right hand side of (4.35) is less than $C_\varepsilon(f)$, for $t \geqslant T_\varepsilon(R)$. This and (4.34) show that

$$S_\varepsilon(t)\,B \subset B_\varepsilon, \qquad \forall t > T_\varepsilon(R)$$

and we achieve the proof of Proposition 4.2. ∎

4.2c. *Lipschitz Properties of F.* We introduce the following cut-off function: $\zeta(x) = 1$ for $x \in [0, 2]$, $\zeta(x) = 3 - x$ for $x \in [2. 33]$ and $\zeta(x) = 0$ for $x \geqslant 3$. And set

$$F(v) = f - \zeta(|v_x|_0^2/2C_1(f))\,\kappa v\,\partial_x v, \tag{4.36}$$

where $C_1(f)$ is given in Proposition 4.2. According to this result, solutions of (4.27)–(4.28) are solutions of

$$\frac{du}{dt} + Au + Cu + F(u) = 0. \tag{4.37}$$

We claim that F in (4.36) satisfies (1.6)–(1.7) with $\alpha = \gamma = 1/2s$. We have already noted that F maps $H_L^1 = D(A^{1/2s})$ into $D(A^0) = H = L_L^2$. We take v and w in $\mathring{H}_L^1$ and write

$$F(v) - F(w) = \zeta(v)\,v\,\partial_x v - \zeta(w)\,w\,\partial_x w$$

where $\bar{\zeta}(v)$ is short hand for $\kappa\zeta(|v_x|^2)/2C_1(f))$. Now

$$|F(v)-F(w)|_0 \leqslant |\bar{\zeta}(v)(v^2-w^2)_x/2|_0 + |(\bar{\zeta}(v)-\bar{\zeta}(w))\, w\,\partial_x w|_0,$$

(we use that $|\zeta(x)-\zeta(y)| \leqslant |x-y|$) $|F(x)-F(w)|_0 \leqslant |\kappa|\ |(v^2-w^2)_x|_0/2 + |K|\ (|v_x|_0^2 - |w_x|_0^2)|\ w\,\partial_x w|_0/(2C_1(f)$. We claim that

$$|F(v)-F(w)|_0 \leqslant C_2\,|\kappa|\ (/f|_0 + \kappa^4\ |f|_0^5)\ |v_x - w_x|_0, \qquad (4.38)$$

where C_2 is a numerical constant. Indeed, if $|v_x|_0^2$ and $|w_x|_0^2$ are both larger than $2C_1(f)$ then $F(v)=F(w)=0$ and (4.38) is obviously satisfied. By symmetry we can assume that $|v_x|_0^2 \leqslant 2C_1(f)$. Then (4.38) follows easily by considering the two cases (i) $|v_x - w_x|_0^2 \geqslant 8C_1(f)$ and (ii) $|v_x - w_x|_0^2 \leqslant 8C_1(f)$.

4.2d. *Construction of Inertial Manifolds.* Thanks to (4.38), we see that (1.6) holds with $\alpha=\gamma=1/2s$. On the other hand the eigenvalues of A: $\Lambda_k = |k|^{2s}$, $k \geqslant 1$ have multiplicity two and the conditions on N in Theorem 1.1 read

$$(N+1)^{2s} \geqslant 3C_2^2\kappa^2(|f|_0 + \kappa^4\ |f|_0^5)^2/2, \qquad (4.39)$$

$$(N+1)^{2s} - N^{2s} \geqslant 30C_2\,|\kappa|\ (|f|_0 + \kappa^4\ |f|_0^5)(N+(N+1)). \qquad (4.40)$$

For $s=1$, (4.40) is not satisfied in general, but for $s>1$ one can always find N satisfying (4.39) and (4.40). Since $(N+1)^{2s} \geqslant N^{2s} + 2sN^{2s-1}$, we deduce that there exists a numerical constant C_3 (independent of s) such that (3.39) and (3.40) are satisfied for

$$2N \geqslant C_3\left(\frac{|\kappa|}{s}(|f|_0 + \kappa^4\ |f|_0^5)\right)^{1/(2s-2)}. \qquad (4.41)$$

The fractional step discretization of (4.37) can be explicited here as follows. We denote by u_k^n the kth Fourier components of u^n and denote by $u_k^n \otimes u_k^n = \sum_{l\in\mathbb{Z}} u_{k-l}^n \bar{u}_l^n$ the kth Fourier component of u^2. Then the fractional step method in Theorem 2.2 reads

$$\begin{aligned}
&(u_k^{n+1/2} - u_k^n)/\gamma + |k|^{2s} u^{n+1/2} + ik\kappa\zeta(|u_x^n|_0^2/2C_1(f))\, u_k^n \otimes u_k^n = 0,\\
&u_k^{n+1} = (1+i\tau(-1)^p k^{2p+1}/2)(1-i\tau(-1)^p k^{2p+1}/2)^{-1} u_k^{n+1/2}, \qquad (4.42)\\
&|u_x^n|_0^2 = \sum_{k\in\mathbb{Z}} |k|^2\,|u_k^n|^2, \qquad k\in\mathbb{Z}.
\end{aligned}$$

This scheme was implicit in the physical variables x and t, in the Fourier variables (4.42) becomes explicit.

Applying Theorems 1.1, 2.2, and 3.1 to the pseudo Korteweg–de Vries–Burgers equations we have the following result.

THEOREM 4.2. *Equation* (4.37) *possesses an M-dimensional manifold in* $\mathring{H}^1_L$, $M = M_\phi$, *where*

$$M \leqslant C_3 \left(\frac{|\kappa|}{s} (|f|_0 + \kappa^4 |f|_0^5) \right)^{1/2s-2)}.$$

For every $\tau > 0$, *satisfying* $\tau C_4^{2s}(|\kappa|/s)(|f|_0 + \kappa^4 |f|_0^5))^{1(s-1)}$, C_3 *and* C_4 *are numerical constants; the discrete iteration* (4.42) *possesses also an M-dimensional inertial manifold in* $\mathring{H}^1_L$, $M_\Gamma = M_{\phi_\tau}$. *Moreover we have the error estimate*

$$\|\phi - \phi_\tau\|_{1/2s} \leqslant K\tau^\zeta(1 + |\text{Log}\, \tau|),$$

where $\zeta = (2s-1)(4p+2-2s)^{-1}$ *for* $s \in]1, p+\frac{1}{2}[$ *and* $\zeta = (2s-1)/(2s)$ *for* $s \geqslant p + \frac{1}{2}$.

4.2e. *Another Modified Equation.* According to Proposition 4.2 and the estimate (4.31), the following balls are absorbing for $\varepsilon > 0$:

$$B_\varepsilon^\infty = \{ v \in H_L^1, \sup_{0 \leqslant x \leqslant L} |v(x)|^4 \leqslant 4(1+\varepsilon) |f|_0^2 C_\varepsilon(f) \}.$$

This suggests to modify the nonlinear term $\kappa u u_x - f - (\zeta(|v|\ \varphi_\infty^{-1})\, kv^2/2)_{x'}$, where ζ is as in Section 4.2c and (we consider B_1^∞)

$$\rho_\infty^4 = 8\ |f|_0^4(1 + 16 C_0 \kappa^8\ |f|_0^8).$$

The function F_∞ is Lipschitz from L_L^2 into H_L^{-1} (the dual space of H_L^1), i.e., from H into $D(A^{-1/2s})$. More precisely, (1.6)–(1.7) holds true with $\alpha = 0$, $\gamma = \frac{1}{2}s$ and

$$L_{F_\infty} \leqslant C_5\ |\kappa|\ (|f|_0 + \kappa^2\ |f|_0^2), \tag{4.40)'}$$

where C_5 is a numerical constant. The corresponding modified $\psi - K - dV$ Burgers equation reads now

$$\frac{du}{dt} + Au + Cu + F_\infty(u) = 0, \tag{4.41)'}$$

and solutions of (4.27)–(4.28) are solutions of (4.41) after a transient. Then applying Theorems 1.1, 2.2, and 3.1, we obtain the analogue of Theorem 4.2. That is exact and approximate inertial manifolds with much smaller dimension M_0 (indeed the relevant cases are when $|\kappa|$ and $|f|_0$ are large),

$$M_0 \leqslant C_6 \left(\frac{|\kappa|}{s} (|f|_0 + \kappa^2\ |f|_0^2) \right)^{1/(2s-2)}.$$

However, these manifolds are imbedded in a larger space, L_L^2 instead of H_L^1 previously. Also, the error estimate holds in that space, i.e., in the norm $\| \|_0$. Finally we note that one cannot compare the two sets of inertial manifolds since they correspond to equations truncated through a different procedure.

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