Existence results for parabolic problems related to fully non linear operators degenerate or singular

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Abstract

In this paper we prove some existence and regularity results concerning parabolic equations

$$u_t = F(x, \nabla u, D^2 u) + f(x,t)$$

with some boundary conditions, on $\Omega\times]0,T[$, where Ω is some bounded domain which possesses the cone property and F is singular or degenerate, with some uniform ellipticity conditions.

1 Introduction

In this paper we consider singular or degenerate parabolic equations of the form

$$\begin{cases} u_t = F(x, \nabla u, D^2 u) + h(x,t)\cdot\nabla u|\nabla u|^\alpha + f(x,t) & \text{in } Q_T \\ u(x,t) = \psi(x,t) & \text{on } \partial Q_T \end{cases}$$

where $Q_T = \Omega\times]0,T[$ and Ω is some bounded domain in $\mathbb{R}^N$ and $T > 0$. Here u_t, ∇u, $D^2 u$ denote respectively the derivative of u with respect to t, the gradient and the hessian with respect to x.

The operator F is uniformly elliptic, homogeneous of degree α, $\alpha > -1$ with respect to the gradient, and positively homogeneous of degree 1 with respect to the Hessian. This operator can be either degenerate ($\alpha > 0$) or singular ($\alpha < 0$).

A typical example is the α+2-Laplace operator, the operators $|\nabla u|^\alpha \mathcal{M}(D^2 u)$, where $\mathcal{M}$ is one of the Pucci's operators, but we also present some non divergence type extension of the $\alpha + 2$ Laplacian.

Our purpose here is to present a convenient and new definition of viscosity solutions for this class of parabolic equations, the difficulty being that since the operator F is not defined on points where the gradient is zero, one cannot "test" such points. In the stationary case [3] this difficulty is overcome by just "not testing" , unless the function be locally constant. Here the situation is more involved and requires some testing.

The existence of solutions for the Dirichlet problem is obtained in three steps. The first one is some comparison theorem, the second is the construction of lower and upper barriers, and finally we use some "Perron's method " adapted to our context, classically obtaining the solution as the supremum of sub-solutions lying between the two barriers. This existence's result is obtained under continuity hypothesis on the data f and ψ.

For the sake of completeness, we establish some Hölder's regularity result, under the assumptions that f is continuous and bounded with respect to x and Hölder's in t, and ψ is Holderian with respect to x and Lipschitz in t.

The case of $\Omega = \mathbb{R}^N$ is also treated, assuming that the data are uniformly continuous bounded.

Analogous problems are studied by Crandall, Kocan, Lions, and Swiech in [9] for the case of Pucci's operators, by Ishii and Souganidis [16] for operators singular or degenerate and homogeneous of degree 1, by Onhuma and Sato [20] in the case of the p-Laplacian, and by Evans and Spruck in [13] for the motion of level sets of mean curvature equations.

In [20] the authors consider the case of the p-Laplacian and a right hand side equals to zero. They give a convenient definition of viscosity solution which provides a comparison principle. This definition requires to introduce a set of admissible test functions when the gradient of u is zero. In [18] , Juutinen and Kawhol treat the case of the infinite Laplacian when the right hand side f is zero and the domain is regular. Let us note that their situation is analogous to the present one when $\alpha = 0$. In their case the operator is linear with respect to $D^2 u$ but it is not well defined on points where the gradient is zero. Though the operator that they consider does not satisfies (H2) one can adapt the definition of viscosity solutions that they propose to our case, and vice versa. That is the reason why we prove in the appendix the equivalence between their solutions and ours, as well as for the solutions in the sense of Ohnuma and Sato.

This paper is organized as follows : In section 2 we give the assumptions on the operator F and we present the notion of viscosity solution which will be adopted in the paper. In section 3 we establish some comparison theorem and exhibit the lower and upper barriers which will allow to get some existence's result in section 4, where we precise the "Perron's" method in the present context. We end the section by establishing some Hölder's regularity results on the solution.

Section 5 is devoted to the case of $\Omega \times \mathbb{R}^+$ and in section 6 we extend our results of existence, uniqueness and regularity to the case of $\mathbb{R}^N \times]0, T[$, under some assumptions of uniform boundedness of the data, f and ψ.

2 Hypothesis and definition of viscosity solutions.

In all that paper, (except in section 6) we shall assume that Ω is some bounded domain which satisfies the uniform exterior cone condition, i.e. we assume that there exist $\phi \in]0, \pi[$ and $\bar{r} > 0$ such that for any $z \in \partial\Omega$ and for an axe through z of direction $\vec{n}_z$,

$$T_\phi = \{x : \frac{(x-z)\cdot \vec{n}_z}{|z-x|} \le \cos\phi\}, \qquad T_\phi \cap \overline{\Omega} \cap B_{\bar{r}}(z) = \{z\}.$$

For a real T positive let $Q_T = \Omega \times]0, T[$. We denote by ∂Q_T the parabolic boundary $(\partial\Omega \times [0, T]) \cup (\overline{\Omega} \times \{0\})$.

Let $\alpha > -1$, $0 < a < A$ be given, let S be the space of symmetric (N, N) matrices, and suppose that F satisfies

(H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$, is continuous with respect to all its variables, and $\forall t \in \mathbb{R}^\star$, $\mu \ge 0$, for all $x \in \Omega$, $p \ne 0$ and $X \in S$,

$$F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X).$$

(H2) For $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \backslash \{0\}$, $M \in S$, $N \in S$, $N \geq 0$

$$a|p|^\alpha tr(N) \leq F(x,p,M+N) - F(x,p,M) \leq A|p|^\alpha tr(N). \qquad (2.1)$$

(H3) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X,Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x,y) \in \mathbb{R}^N$, $x \neq y$

$$F(x, \zeta(x-y), X) - F(y, \zeta(x-y), -Y) \leq \omega(\zeta|x-y|^2).$$

Sometimes this condition $(H3)$ can be replaced by the weaker assumption, which is sufficient to prove Hölder's regularity results :

(H4) There exists a continuous function $\tilde{\omega}$, $\tilde{\omega}(0) = 0$ such that for all x, y, in Ω, $p \neq 0$, $\forall X \in S$

$$|F(x,p,X) - F(y,p,X)| \leq \tilde{\omega}(|x-y|)|p|^\alpha |X|.$$

We assume that h is continuous and bounded on Q_T with values in $\mathbb{R}^N$ and satisfies (H5) :

There exists $\omega_h \leq 1$ and $c_h > 0$ such that for all (x,t), (x,s) in Q_T

$$|h(x,t) - h(x,s)| \leq c_h |t-s|^{\omega_h}.$$

Furthermore

- Either $\alpha \leq 0$ and for all (x,y) in Ω and $t \in]0,T[$

$$|h(x,t) - h(y,t)| \leq c_h |x-y|^{1+\alpha}$$

- or $\alpha > 0$ and for all (x,y) in Ω and $t \in]0,T[$

$$(h(x,t) - h(y,t) \cdot x - y) \leq 0.$$

Examples

1) Let $0 < a < A$ be given and $\mathcal{M}^+_{a,A}$ be the Pucci's operator

$$\mathcal{M}^+_{a,A}(N) = Atr(N^+) - atr(N^-)$$

where $N^\pm$ denotes the positive and negative parts of the matrix N, $\mathcal{M}^-_{a,A}(N) = -\mathcal{M}^+_{a,A}(-N)$. Then F defined as

$$F(x,p,M) = |p|^\alpha(\mathcal{M}^\pm_{a,A}(M)$$

satisfies (H1), (H2).

2) Let $\alpha > -1$, then the $\alpha + 2$-Laplace operator

$$F(p,M) = |p|^\alpha \left(tr(M) + \alpha\langle M\frac{p}{|p|}, \frac{p}{|p|}\rangle \right)$$

satisfies (H1- and (H2).

3) Suppose that $1 < q \leq 2$, $c(q) \geq q - 2$, suppose that B is some invertible Lipschitz function which sends $\overline{\Omega}$ into S, and define

$$F(x,p,N) = |p|^{q-2}tr(B^2(x)N) + c(q)|p|^{q-4}(N(B(x)p, B(x)p))$$

Then F satisfies (H1), (H2), (H3). (see [3]).

Concerning the right hand side f we shall assume that it is at least continuous and will precise further regularity when it will be needed.

We now give the definition of viscosity solutions adapted to our context.

It is well known that when dealing with viscosity respectively sub- and super- solutions one works with

$$u^\star(x,t) = \limsup_{(y,\tau),|(y,\tau)-(x,t)|\leq r} u(y,\tau)$$

and

$$u_\star(x,t) = \liminf_{(y,\tau),|(y,\tau)-(x,t)|\leq r} u(y,\tau).$$

It is easy to see that $u_\star \leq u \leq u^\star$ and $u^\star$ is upper semicontinuous (USC), $u_\star$ is lower semicontinuous (LSC). See e.g. [8, 14].

In the sequel $\mathcal{C}^{1,2}$ denotes the space of functions which are $\mathcal{C}^1$ in the time variable, and $\mathcal{C}^2$ in space.

Definition 1 *We shall say that u, locally bounded, is a viscosity sub-solution of*

$$u_t - F(x, \nabla u, D^2 u) - h(x,t) \cdot \nabla u |\nabla u|^\alpha \leq f(x,t) \quad in \quad \Omega \times (0,T)$$

if , for any $(\bar{x}, \bar{t}) \in \Omega \times (0,T)$,

- *either for all* $\varphi \in \mathcal{C}^{1,2}$ *such that* $(u^\star - \varphi)$ *has a local maximum on* $(\bar{x}, \bar{t})$ *with* $\nabla \varphi(\bar{x}, \bar{t}) \neq 0$

$$\varphi_t(\bar{x}, \bar{t}) - F(\bar{x}, \nabla \varphi(\bar{x}, \bar{t}), D^2 \varphi(\bar{x}, \bar{t})) - h(\bar{x}, \bar{t}) \cdot \nabla \varphi |\nabla \varphi|^\alpha (\bar{x}, \bar{t}) \leq f(\bar{x}, \bar{t}).$$

- *or, if there exists* δ_1 *and* $\varphi \in \mathcal{C}^1(]\bar{t} - \delta_1, \bar{t} + \delta_1[)$, *such that for any* $t \in]\bar{t} - \delta_1, \bar{t} + \delta_1[$

$$\begin{cases} u^\star(\bar{x}, \bar{t}) - \varphi(\bar{t}) \geq u^\star(\bar{x}, t) - \varphi(t) \\ \sup_{t \in]\bar{t} - \delta_1, \bar{t} + \delta_1[} (u^\star(x,t) - \varphi(t)) \text{ is constant in a neighborhood of } \bar{x}, \end{cases}$$

then

$$\varphi'(\bar{t}) \leq f(\bar{x}, \bar{t}).$$

u, locally bounded, is a viscosity super-solution of

$$u_t - F(x, \nabla u, D^2 u) - h(x,t) \cdot \nabla u |\nabla u|^\alpha \geq f(x,t) \quad in \quad \Omega \times (0,T)$$

if, for any $(\bar{x}, \bar{t}) \in \Omega \times (0,T)$,

- *either for all* $\varphi \in \mathcal{C}^{1,2}$ *such that* $(u_\star - \varphi)$ *has a local minimum on* $(\bar{x}, \bar{t})$ *with* $\nabla \varphi(\bar{x}, \bar{t}) \neq 0$,

$$\varphi_t(\bar{x}, \bar{t}) - F(\bar{x}, \nabla \varphi(\bar{x}, \bar{t}), D^2 \varphi(\bar{x}, \bar{t})) - h(\bar{x}, \bar{t}) \cdot \nabla \varphi |\nabla \varphi|^\alpha (\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).$$

- *or, if there exists* δ_1 *and* $\varphi \in \mathcal{C}^1(]\bar{t} - \delta_1, \bar{t} + \delta_1[)$ *such that for any* $t \in]\bar{t} - \delta_1, \bar{t} + \delta_1[$

$$\begin{cases} u_\star(\bar{x}, \bar{t}) - \varphi(\bar{t}) \leq u_\star(\bar{x}, t) - \varphi(t) \\ \inf_{t \in]\bar{t} - \delta_1, \bar{t} + \delta_1[} (u_\star(x,t) - \varphi(t)) \text{ is locally constant in a neighborhood of } \bar{x}, \end{cases}$$

then

$$\varphi'(\bar{t}) \leq f(\bar{x}, \bar{t}).$$

Finally a locally bounded function u is a viscosity solution when u is both a viscosity sub- and super-solution.

Remark 1 *We prove in the appendix that the solutions in the sense of definition 1are the same as those of Onhuma and Sato in the case where $\alpha \neq 0$, and to those of Juutinen and Kawohl in the case of the infinity Laplacian.*

Remark 2 *For convenience of the reader we recall the definition of semi-jets for parabolic problems :*

$$\begin{aligned} J^{2,+}u(\bar{x},\bar{t}) &= \{(q,p,X) \in \mathbb{R} \times (\mathbb{R}^N)^\star \times S,\ q(t-\bar{t}) + p.(x-\bar{x}) \\ &+ \frac{1}{2}(x-\bar{x})^T X(x-\bar{x}) + o(|t-\bar{t}|) + o(|x-\bar{x}|^2) \geq u(x,t) - u(\bar{x},\bar{t})\} \end{aligned}$$

and

$$\begin{aligned} J^{2,-}u(\bar{x},\bar{t}) &= \{(q,p,X) \in \mathbb{R} \times (\mathbb{R}^N)^\star \times S,\ q(t-\bar{t}) + p.(x-\bar{x}) \\ &+ \frac{1}{2}(x-\bar{x})^T X(x-\bar{x}) + o(|t-\bar{t}|) + o(|x-\bar{x}|^2) \leq u(x,t) - u(\bar{x},\bar{t})\} \end{aligned}$$

and the closed semi-jets

$$\begin{aligned} \bar{J}^{2,+}u(\bar{x},\bar{t}) &= \{(q,p,X) \in \mathbb{R} \times (\mathbb{R}^N)^\star \times S, \exists\ (x_n,t_n) \to (\bar{x},\bar{t}) \text{ and} \\ & (q_n,p_n,X_n) \in J^{2,+}u(x_n,t_n),\ (u(x_n,t_n),q_n,p_n,X_n) \to (u(\bar{x},\bar{t}),q,p,X)\} \end{aligned}$$

$$\begin{aligned} \bar{J}^{2,-}u(\bar{x},\bar{t}) &= \{(q,p,X) \in \mathbb{R} \times (\mathbb{R}^N)^\star \times S, \exists\ (x_n,t_n) \to (\bar{x},\bar{t}) \text{ and} \\ & (q_n,p_n,X_n) \in J^{2,-}u(x_n,t_n),\ (u(x_n,t_n),q_n,p_n,X_n) \to (u(\bar{x},\bar{t}),q,p,X)\}. \end{aligned}$$

It is classical that one can deal with closed semi jets in place of semi jets or test functions, as it can be seen from [14], [8].

In the following we shall denote by $1_{\{f\}}$ the equation

$$u_t = F(x,\nabla u, D^2 u) + h(x,t)\cdot \nabla u|\nabla u|^\alpha + f(x,t)$$

and for ψ a continuous function defined on ∂Q_T, by $1_{\{f,\psi\}}$ the boundary value problem

$$\begin{cases} u_t = F(x,\nabla u, D^2 u) + h(x,t)\cdot \nabla u|\nabla u|^\alpha + f(x,t) & \text{in } Q_T \\ u(x,t) = \psi(x,t) & \text{on } \partial Q_T \end{cases}$$

Remark 3 *Let us note that if u is a sub-solution (respectively super-solution) of $1_{\{f\}}$ and if φ is some $\mathcal{C}^1$ function depending only on t, $(x,t) \mapsto u(x,t) + \varphi(t)$ is a sub-solution (respectively super-solution) of $1_{\{f+\varphi'\}}$.*

3 Comparison principle and barriers.

In all this section we assume that Ω is some bounded domain which satisfies the uniform exterior cone condition, that F satisfies (H1), (H2), (H3), and h satisfies (H5).

We begin to prove some comparison principle for the operator $u_t - F(x, \nabla u, D^2u) - h(x,t)\cdot\nabla u|\nabla u|^\alpha$. One of its consequences is the uniqueness of the solutions for $1_{\{f,\psi\}}$.

Theorem 1 *Suppose that u is a sub-solution bounded for $1_{\{g\}}$ and v is a super solution bounded of $1_{\{f\}}$ with $g \leq f$ in $\Omega\times]0,T[$, g being upper semicontinuous and f being lower semicontinuous. Suppose that $u^\star \leq v_\star$ on $(\partial\Omega\times[0,T])\cup(\overline{\Omega}\times\{0\})$, then $u^\star \leq v_\star$ in $\Omega\times]0,T)$.*

The proof of this theorem requires the following technical lemma whose proof is postponed after the proof of theorem 1 for the sake of clearness.

Lemma 1 *Suppose that Ω is some open set and $0 \in \Omega$. Suppose that u is a super-solution of*

$$u_t - F(x, \nabla u, D^2u) - h(x,t)\cdot\nabla u|\nabla u|^\alpha \geq f(x,t)$$

in $Q_T = \Omega\times]0,T[$ and suppose that C_1 is some constant , that φ is some $\mathcal{C}^1$ function on $]0,T[$, that $k > \sup(2, \frac{\alpha+2}{\alpha+1})$ and $(0,\bar{t}) \in \Omega\times]0,T[$ are such that for some $\delta_1 > 0$

$$\inf_{x\in B(0,\delta_1),|t-\bar{t}|<\delta_1} (u(x,t) - \varphi(t) + C_1|x|^k) = u(0,\bar{t})$$

Then

$$\varphi'(\bar{t}) \geq f(0,\bar{t}).$$

Proof of theorem 1 :

Suppose by contradiction that $u(\bar{x},\bar{t}) > v(\bar{x},\bar{t})$ for some $(\bar{x},\bar{t}) \in Q_T$, let $\kappa > 0$ be such that

$$\frac{2\kappa}{T-\bar{t}} < \frac{(u-v)(\bar{x},\bar{t})}{2},$$

then $(x,t) \mapsto u(x,t) - \frac{\kappa}{T-t}$ is a strict sub-solution, $(x,t) \mapsto v(x,t) + \frac{\kappa}{T-t}$ is a strict super-solution and $u_1 - v_1 > 0$ somewhere in Q_T. Moreover the maximum of $u_1 - v_1$ cannot be achieved in $|t-T| < \frac{T-\bar{t}}{2}$, since in that set one has

$$u - \frac{K}{T-t} - (v + \frac{K}{T-t}) \leq \sup(u-v) - \frac{4\kappa}{T-\bar{t}}$$

while

$$u(\bar{x},\bar{t}) - \frac{K}{T-\bar{t}} - (v(\bar{x},\bar{t}) + \frac{K}{T-\bar{t}}) \geq \sup(u-v) - \frac{2\kappa}{T-\bar{t}}.$$

In the following we replace u by $u - \frac{\kappa}{T-t}$ which is a sub-solution of $1_{f-\frac{\kappa}{(T-t)^2}}$ and v by $v + \frac{\kappa}{T-t}$ which a super-solution of $1_{f+\frac{\kappa}{(T-t)^2}}$.

We define for $j \in \mathbf{N}$ and for $k > \sup(2, \frac{\alpha+2}{\alpha+1}, \frac{2(1+\alpha)}{\omega_h})$,

$$\Psi_j(x,t,y,s) = u^\star(x,t) - v_\star(y,s) - \frac{j}{2}|t-s|^2 - \frac{j}{k}|x-y|^k$$

Then ψ_j achieves its maximum on $(x_j,t_j,y_j,s_j) \in (\Omega \times]0,T[)^2$. It is classical that the sequences (x_j,t_j), and (y_j,s_j) both converge to $(\bar{x},\bar{t})$ which is a maximum point for $u^\star - v_\star$, and that $j|s_j-t_j|^2 + j|x_j-y_j|^k \to 0$.

We want to prove that for j large enough $x_j \neq y_j$. Suppose not i.e. $x_j = y_j$ then

$$(y,s) \mapsto v_\star(x_j,s_j) - \frac{j}{k}|x_j-y|^k - \frac{j}{2}|s-t_j|^2 + \frac{j}{2}|t_j-s_j|^2$$

would be a test function from below for $v_\star$ at (x_j,s_j). Then applying Lemma 1 in its form for super-solutions with $C_1 = \frac{j}{k}$, φ replaced by $t \mapsto v_\star(x_j,s_j) - \frac{j}{2}|t-t_j|^2 + \frac{j}{2}|t_j-s_j|^2$, replacing 0 by x_j, and $\bar{t}$ by s_j one would get that

$$-j(s_j-t_j) \geq \frac{\kappa}{T^2} + f(x_j,s_j).$$

On the other hand

$$(x,t) \mapsto u^\star(x_j,t_j) + \frac{j}{k}|x_j-x|^k + \frac{j}{2}|t-s_j|^2 - \frac{j}{2}|t_j-s_j|^2$$

would be a test function from above for $u^\star$ on (x_j,t_j). Using Lemma 1 in its form for sub-solutions, with φ replaced by $t \mapsto u(x_j,t_j) + \frac{j}{2}|t-t_j|^2 - \frac{j}{2}|t_j-s_j|^2$ 0 by x_j, C_1 by $-\frac{j}{k}$, one gets that

$$j(t_j-s_j) \leq g(x_j,t_j) - \frac{\kappa}{T^2}.$$

Substracting the two inequalities, passing to the limit and using the upper semicontinuity of g and the lower semicontinuity of f, one gets that

$$\lim_{j\to+\infty} j(t_j-s_j) + j(s_j-t_j) \leq -\frac{2\kappa}{T^2} + \limsup_{j\to+\infty}(g(x_j,t_j) - f(x_j,s_j)) \leq -\frac{2\kappa}{T^2}$$

which is a contradiction.

We have then proved that $x_j \neq y_j$.

By Ishii's lemma [8], (see also lemma 2.1 in [3]) there exist $(X_j, Y_j) \in S^2$, with

$$(j(t_j - s_j), j|x_j - y_j|^{k-2}(x_j - y_j), X_j) \in \bar{J}^{2,+}u^\star(x_j, t_j)$$

$$(j(t_j - s_j), j|x_j - y_j|^{k-2}(x_j - y_j), -Y_j) \in \bar{J}^{2,-}v_\star(y_j, s_j)$$

and for some positive constant c

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq cj|x_j - y_j|^{k-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

This implies that, using assumption (H3) and the fact that $j|x_j - y_j|^k \to 0$

$$\begin{aligned} \frac{\kappa}{T^2} + f(y_j, s_j) \;&\leq\; j(t_j - s_j) - F(y_j, j|x_j - y_j|^{k-2}(x_j - y_j), -Y_j) \\ &-\; j^{1+\alpha}h(y_j, s_j) \cdot (x_j - y_j)|x_j - y_j|^{k-2+(k-1)\alpha} \\ &\leq\; j(t_j - s_j) - F(x_j, j|x_j - y_j|^{k-2}(x_j - y_j), X_j) + o(1) \\ &-\; j^{1+\alpha}h(x_j, t_j) \cdot (x_j - y_j)|x_j - y_j|^{k-2+(k-1)\alpha} + o(1) \\ &\leq\; g(x_j, t_j) - \frac{\kappa}{T^2} + o(1). \end{aligned}$$

In the previous inequalities we have used

$$\begin{aligned} |h(x_j, t_j) - h(x_j, s_j)| \quad & |x_j - y_j|^{(k-1)(1+\alpha)} j^{1+\alpha} \\ \leq\; & c_h|t_j - s_j|^{\omega_h} j^{1+\alpha}|x_j - y_j|^{(k-1)(1+\alpha)} \\ \leq\; & (j|t_j - s_j|^2)^{\frac{\omega_h}{2}} (j|x_j - y_j|^k)^{\frac{(1+\alpha)(k-1)}{k}} j^{\frac{1+\alpha}{k} - \frac{\omega_h}{2}} \\ =\; & o(1) \end{aligned}$$

and when $\alpha < 0$

$$|h(x_j, s_j) - h(y_j, s_j)||x_j - y_j|^{(k-1)(1+\alpha)} j^{1+\alpha} \leq j^{1+\alpha}|x_j - y_j|^{k(1+\alpha)} = o(1).$$

Using the lower semicontinuity of f, the uppersemicontinuity of g and letting $j \to +\infty$ we get a contradition.

Proof of Lemma 1 First replacing if necessary φ by $\varphi(t) - C_2|t - \bar{t}|^2$ for some constant $C_2 > 0$ and C_1 by some constant $> C_1$ one can assume that the infimum is strict in x and t separately.

Clearly $\psi(x,t) = \varphi(t) - C_1|x|^k - C_2(t-\bar{t})^2$ is a test function for u in $(0,\bar{t})$ but its gradient with respect to x is zero. So we are going to prove that either the function $t \mapsto \varphi(t) - C_2|t-\bar{t}|^2$ is a " test" function as in the second case of the definition of viscosity super-solution and then the conclusion of the Lemma is immediate. Or, if this is not the case, then it is possible to construct a sequence of points tending to $(0,\bar{t})$ for which there exists a test function which gradient with respect to x is different from zero, but tend to zero. Then passing to the limit we get the required inequality.

Hence we suppose first that the function $t \mapsto \varphi(t) - C_2|t-\bar{t}|^2$ is as in the definition of viscosity super-solution i.e. we suppose that there exists $\delta_1 > 0$, and $\bar{\delta} > 0$ such that for all $x \in B(0,\bar{\delta})$,

$$\inf_{|t-\bar{t}|<\delta_1} \{v(x,t) - \varphi(t) + C_2(t-\bar{t})^2\} = \inf_{|t-\bar{t}|<\delta_1} \{v(0,t) - \varphi(t) + C_2(t-\bar{t})^2\}.$$

We claim that this infimum is achieved on $(0,\bar{t})$. Indeed, the infimum is less or equal to $v(0,\bar{t})$ and on the other hand it is more than $\inf_{x\in B(0,\delta_1),|t-\bar{t}|<\delta_1}\{v(x,t) + C_1|x|^k - \varphi(t) + C_2(t-\bar{t})^2\}$ which equals $v(0,\bar{t})$.

Then the conclusion given in that case in the definition of viscosity super-solution is that $\varphi'(\bar{t}) \geq f(0,\bar{t})$.

We now suppose that we are not in this situation i.e. that $x \mapsto \inf_{|t-\bar{t}|<\delta_1} v(x,t) - \varphi(t) + C_2|t-\bar{t}|^2$ is not constant in a neighborhood of 0.

Recall that since the infimum is strict in x and t separately, for all $\delta > 0$, $\delta < \delta_1$ there exists $\epsilon(\delta) > 0$ such that

$$\begin{aligned} \inf\Bigg(\inf_{|t-\bar{t}|>\delta, x\in B(0,\delta_1)} \{v(x,t) + {} & C_1|x|^k - \varphi(t) + C_2(t-\bar{t})^2\}, \\ & \inf_{|t-\bar{t}|\leq\delta_1, |x|>\delta} \{v(x,t) + C|x|^k - \varphi(t) + C_2(t-\bar{t})^2\} \Bigg) \\ \geq {} & v(0,\bar{t}) + \epsilon(\delta). \end{aligned}$$

We now choose $\delta_2 \leq \inf(\frac{\epsilon(\delta)}{4C_1 k(2\delta_1)^{k-1}}, \delta)$. Then, with that choice, for all $x \in B(0,\delta_2)$

$$\inf_{y\in B(0,\delta_1),|t-\bar{t}|\leq\delta_1} \{v(y,t) + C_1|x-y|^k - \varphi(t) + C_2(t-\bar{t})^2\} \leq v(0,\bar{t}) + \frac{\epsilon(\delta)}{4}$$

while

$$\inf_{|y|>\delta,|t-\bar{t}|\leq\delta_1} \{v(y,t) - \varphi(t) + C_1|y-x|^k + C_2(t-\bar{t})^2\} \geq v(0,\bar{t}) + \frac{3\epsilon(\delta)}{4}.$$

Moreover one also has

$$\begin{aligned}
&\inf_{y\in B(0,\delta_1),|t-\bar t|>\delta}\{v(y,t)-\varphi(t)+C|x-y|^k+C_2(t-\bar t)^2\}\\
\geq\ & v(0,\bar t)+\frac{3\epsilon(\delta)}{4}.
\end{aligned}$$

This implies that for all $x\in B(0,\delta_2)$

$$\begin{aligned}
&\inf_{y\in B(0,\delta_1),|t-\bar t|<\delta_1}\{v(y,t)+C_1|y-x|^k-\varphi(t)+C_2(t-\bar t)^2\}\\
=&\inf_{y\in B(0,\delta),|t-\bar t|\leq\delta}\{v(y,t)+C_1|y-x|^k-\varphi(t)+C_2(t-\bar t)^2\}. \qquad (3.2)
\end{aligned}$$

Since $x\mapsto \inf_{|t-\bar t|<\delta_1}\{v(x,t)-\varphi(t)+C_2|t-\bar t|^2\}$ is not constant in a neighborhood of 0, there exist $(x_\delta,y_\delta)\in B(0,\delta_2)$

$$\inf_{|t-\bar t|<\delta_1}\{v(x_\delta,t)-\varphi(t)+C_2|t-\bar t|^2\}>\inf_{|t-\bar t|<\delta_1}\{v(y_\delta,t)-\varphi(t)+C_2|t-\bar t|^2\}+C_1|x_\delta-y_\delta|^k\}$$

Hence

$$\inf_{y\in B(0,\delta_1),|t-\bar t|<\delta_1}\{v(y,t)-\varphi(t)+C_1|x_\delta-y|^k+C_2|t-\bar t|^2\}$$

is achieved on some point (z_δ,t_δ) with $z_\delta\neq x_\delta$. Indeed if it was achieved on (x_δ,t_δ) for some t_δ one would have

$$\begin{aligned}
v(x_\delta,t_\delta)-\varphi(t_\delta)\ &+\ C_2|t_\delta-\bar t|^2\\
&=\inf_{y\in B(0,\delta_1),|t-\bar t|<\delta_1}\{v(y,t)-\varphi(t)+C_1|x_\delta-y|^k+C_2|t-\bar t|^2\}\\
&\leq \inf_{|t-\bar t|<\delta_1}\{v(y_\delta,t)-\varphi(t)+C_1|y_\delta-x_\delta|^k+C_2|t-\bar t|^2\}\\
&< \inf_{|t-\bar t|<\delta_1}\{v(x_\delta,t)-\varphi(t)+C_2|t-\bar t|^2\}\\
&\leq v(x_\delta,t_\delta)-\varphi(t_\delta)+C_2|t_\delta-\bar t|^2,
\end{aligned}$$

a contradiction. Moreover using (3.2), the infimum is achieved in $B(0,\delta)\times]\bar t-\delta,\bar t+\delta[$.

All this implies that $(y,t)\mapsto v(z_\delta,t_\delta)+\varphi(t)-\varphi(t_\delta)+C_1|x_\delta-z_\delta|^k-C_1|x_\delta-y|^k+C_2(t_\delta-\bar t)^2-C_2|t-\bar t|^2$ is a test function for v on (z_δ,t_δ) and since v is a super-solution

$$\begin{aligned}
\varphi'(t_\delta)-2C_2(t_\delta-\bar t)\ &-\ F(-C_1k|x_\delta-z_\delta|^{k-2}(z_\delta-x_\delta),X_\delta)\\
&+\ C_1^{1+\alpha}k^{1+\alpha}|x_\delta-z_\delta|^{(k-1)(\alpha+1)-1}h(z_\delta,t_\delta)\cdot(z_\delta-x_\delta)\\
&\geq\ f(z_\delta,t_\delta)
\end{aligned}$$

where $X_\delta = -D^2(C_1|x_\delta - y|^k)\,|_{y=z_\delta}$. We have finally obtained that

$$\varphi'(t_\delta) - 2C_2(t_\delta - \bar{t}) + C_1^{1+\alpha}\left(|x_\delta - z_\delta|^{k(\alpha+1)-\alpha-2} + |h|_\infty k^{1+\alpha}(2\delta)^{(k-1)(\alpha+1)}\right) \geq f(z_\delta, t_\delta).$$

Using $x_\delta \in B(0,\delta_2) \subset B(0,\delta)$, $z_\delta \in B(0,\delta)$ and $k > \frac{\alpha+2}{\alpha+1}$,

$$\varphi'(t_\delta) + o(1) \geq f(z_\delta, t_\delta).$$

Letting δ go to zero, and using the lower semicontinuity of f one gets the result. This ends the proof of lemma 1.

We now construct a super- solution and a sub-solution for $1_{\{f,\psi\}}$ We recall that in [7] we constructed a global barrier for the stationary case:

Proposition 1 *For all $z \in \partial\Omega$, there exists some function W_z continuous on $\overline{\Omega}$, such that $W_z(z) = 0$, $W_z > 0$ in $\Omega \setminus \{z\}$, which satisfies*

$$F(x, \nabla W_z, D^2 W_z) + h(x,t)\cdot \nabla W_z |\nabla W_z|^\alpha \leq -1 \quad in \quad \Omega.$$

Furthermore $\nabla W_z \neq 0$ everywhere and there exist $\underline{c} > 0$, $\overline{c} > 0$ and $\gamma \in]0,1[$ which depend on the parameters of the cone, such that for all $z \in \partial\Omega$ and $x \in \Omega$

$$\underline{c}|z - x|^\gamma \leq W_z(x) \leq \overline{c}|z - x|^\gamma.$$

Remark 4 *One can ask, up to change the constants γ and the constants $\underline{c}$ and $\overline{c}$ that W_z be such that $-W_z$ be a sub-solution of the equation*

$$F(x, \nabla V, D^2 V) + h(x,t)\cdot \nabla V|\nabla V|^\alpha \geq 1 \quad in \quad \Omega.$$

The proof of Proposition 1 can be found in [7].

We now give some existence's result of super-solutions and sub-solutions for the parabolic problem.

Proposition 2 *Suppose that ψ is continuous on ∂Q_T and that f is uniformly bounded. Then there exists a continuous super-solution W of $1_{\{|f|_\infty,\psi\}}$.*

Under the same assumptions there exists a continuous sub-solution V of $1_{\{-|f|_\infty,\psi\}}$.

Proof of proposition 2.

We prove the proposition assuming that ψ is Hölder's continuous of exponent γ in space and Lipschitz in time, because we provide in that case some expression more explicit which will be needed for the Hölder's regularity of solutions proved later. We shall give the changes to bring when ψ is only continuous at the end of this proof.

Let c_ψ be some holder's constant in space for ψ. We define

$$W_1(x,t) := \inf_{(z,\tau)\in\partial\Omega\times[0,T]} \{\psi(z,\tau) + \left(\frac{c_\psi}{\underline{c}} + (|\psi_t|_\infty + |f|_\infty)^{\frac{1}{1+\alpha}}\right) W_z(x) + |\psi_t|_\infty |t-\tau|\}.$$

Let us note that $\left(\frac{c_\psi}{\underline{c}} + (|\psi_t|_\infty + |f|_\infty)^{\frac{1}{1+\alpha}}\right) W_z(x) + |\psi_t|_\infty|t-\tau|$ is a super-solution of $1_{\{|f|_\infty\}}$ since defining $\lambda_2 = \frac{c_\psi}{\underline{c}} + (|\psi_t|_\infty + |f|_\infty)^{\frac{1}{1+\alpha}}$, one has $\lambda_2 > \lambda_1 = (|\psi_t|_\infty + |f|_\infty)^{\frac{1}{1+\alpha}}$ and then

$$\begin{aligned} -F(x, \lambda_2\nabla W_z, \quad & \lambda_2 D^2 W_z) - h(x)\cdot\lambda_2\nabla W_z|\lambda_2\nabla W_z|^\alpha \\ = \quad & -\left(\frac{\lambda_2}{\lambda_1}\right)^{1+\alpha} \left(F(x,\lambda_1\nabla W_z, \lambda_1 D^2 W_z) + h(x,t)\cdot\nabla(\lambda_1 W_z)|\nabla(\lambda_1 W_z)|^\alpha\right) \\ \geq \quad & -F(x,\lambda_1 DW_z, \lambda_1 D^2 W_z) - h(x,t)\cdot\nabla(\lambda_1 W_z)|\nabla(\lambda_1 W_z)|^\alpha \\ \geq \quad & |f|_\infty + |\psi_t|_\infty \end{aligned}$$

Moreover in the viscosity sense , $\partial_t(|t-\tau|) \geq -1$. This implies that all the functions in the infimum are super-solutions of $1_{\{|f|_\infty\}}$.

Since all the super-solutions in the infimum above have their gradient with respect to $x \neq 0$ everywhere, one can apply the classical result of [8], [14] on the infimum of super-solutions to obtain that W_1 is a super-solution.

We prove that W_1 satisfies the boundary condition on the lateral boundary $W_1(x,t) := \psi(x,t)$ for $x \in \partial\Omega$ and $t \in [0,T]$. Indeed first taking (x,t) in the infimum one gets $W_1(x,t) \leq \psi(x,t)$. On the other hand for all $(z,\tau) \in \partial\Omega\times[0,T]$ $\psi(z,\tau) + \frac{c_\psi}{\underline{c}}\underline{c}|x-z|^\gamma + |\psi_t|_\infty|t-\tau| \geq \psi(x,t)$ which implies by considering the infimum, the reverse inequality.

The same arguments permit to check that $W_1(x,0) \geq \psi(x,0)$ for all $x \in \Omega$.

We now define $q_1 = \sup\{2, \frac{\alpha+2}{\alpha+1}\}$, $q = \frac{q_1}{\gamma}$, $c_q = q(q-1)^{\frac{1-q}{q}}$, and also

$$K_2 = (\text{diam }\Omega|h|_\infty + A(N+q_1-2))(\text{diam }\Omega)^{\sup(\alpha,0)}. \tag{3.3}$$

Then, it is not difficult to see that for any positive constant K_1 and for all $y \in \overline{\Omega}$

$$(x,t) \mapsto K_1|x-y|^{q_1} + K_1^{1+\alpha}K_2 t$$

is a super-solution of $1_{\{0\}}$ and then in particular taking $K_1 = \frac{c_\psi^q}{c_q^q \kappa^{q-1}}$ with c_q defined above, for all $\kappa \in \mathrm{I\!R}^+$ and $y \in \Omega$

$$(x,t) \mapsto \frac{c_\psi^q}{c_q^q \kappa^{q-1}}|x-y|^{q_1} + (|f|_\infty + |\psi_t|_\infty)t + \left(\frac{c_\psi^q}{c_q^q \kappa^{q-1}}\right)^{1+\alpha} K_2 t$$

is a super-solution of $1_{\{|f|_\infty\}}$.

Then if we define

$$W_2(x,t) := \inf_{y\in\overline{\Omega},\kappa\in\mathrm{I\!R}^+}\{\psi(y,0)+\kappa+\frac{c_\psi^q}{c_q^q \kappa^{q-1}}|x-y|^{q_1}+(|f|_\infty+|\psi_t|_\infty)t+\left(\frac{c_\psi^q}{c_q^q \kappa^{q-1}}\right)^{1+\alpha} K_2 t\},$$

W_2 being the infimum of super-solutions of $1_{|f|_\infty}$, using proposition 3 later in its form for super-solutions, it is a super-solution of $1_{|f|_\infty}$.

We need to check that $W_2(x,0) = \psi(x)$. On one hand, by taking $y = x$ in the infimum and $t = 0$ one gets $W_2(x,0) \le \kappa + \psi(x,0)$ for all κ and on the second hand, we use the identity for $q > 1$, and for any positive number P

$$\inf_{\kappa\in\mathrm{I\!R}^+}\{\kappa + \frac{P}{c_q^q \kappa^{q-1}}\} = P^{\frac{1}{q}} \tag{3.4}$$

that we apply here with $P = c_\psi^q|x-y|^{q_1}$. It gives

$$\begin{aligned} W_2(x,0) &= \inf_{y\in\overline{\Omega},\kappa\in\mathrm{I\!R}^+}\{\psi(y,0) + \kappa + \frac{c_\psi^q}{c_q^q \kappa^{q-1}}|x-y|^{q_1}\} \\ &= \inf_{y\in\overline{\Omega}}\{\psi(y,0) + c_\psi|x-y|^\gamma\} \\ &\ge \psi(x,0). \end{aligned}$$

We need also to check that $W_2(x,t) \ge \psi(x,t)$ when $x \in \partial\Omega$. For that aim we use for all $x \in \overline{\Omega}$

$$\begin{aligned} W_2(x,t) \ge \inf_{y\in\overline{\Omega},\kappa\in\mathrm{I\!R}^+}\{\psi(y,0) + \kappa + \frac{c_\psi^q}{c_q^q \kappa^{q-1}}|x-y|^{q_1}\} + |\psi_t|_\infty|t| &= \psi(x,0) + |\psi_t|_\infty|t| \\ &\ge \psi(x,t). \end{aligned}$$

We now define

$$W(x,t)=\inf(W_1(x,t),W_2(x,t))$$

Then W is a super-solution of $1_{\{|f|_\infty,\psi\}}$

Similarly one can define a sub-solution :

$$V(x,t)=\sup(V_1(x,t),V_2(x,t))$$

with

$$V_1(x,t):=\sup_{(z,\tau)\in\partial\Omega\times[0,T],\kappa\in\mathbb{R}^+}\{\psi(z,\tau)-\left(\frac{c_\psi}{\underline{c}}+(|\psi_t|_\infty+|f|_\infty)^{\frac{1}{1+\alpha}}\right)W_z(x)-|\psi_t|_\infty|t-\tau|\}.$$

and

$$V_2(x,t)=\sup_{y\in\overline{\Omega},\kappa\in\mathbb{R}^+}\{\psi(y,0)-\kappa-\frac{c_\psi^q}{c_q^q\kappa^{q-1}}|x-y|^{q_1}-(|f|_\infty+|\psi_t|_\infty)t-\left(\frac{c_\psi^q}{c_q^q\kappa^{q-1}}\right)^{1+\alpha}K_2t\}$$

and K_2 has been defined before.Then V is a sub-solution of $1_{\{-|f|_\infty,\psi\}}$.

We now give the changes to bring when ψ is only continuous. The following construction is similar to the constrution in [9].

Let ω_ψ^t and ω_ψ^x be the modulus of continuity of ψ respectively in t and x, uniformly in the other variable, i.e.

$$\omega_\psi^t(\delta)=\sup_{\{t,\ \tau,|t-\tau|\le\delta,\ x\in\overline{\Omega}\}}|\psi(x,t)-\psi(x,\tau)|$$

and

$$\omega_\psi^x(\delta)=\sup\left(\sup_{\{t\in[0,T],(y,z)\in\partial\Omega,\ |y-z|\le\delta\}}|\psi(y,t)-\psi(z,t)|,\sup_{\{(y,z)\in\overline{\Omega},\ |y-z|\le\delta\}}|\psi(y,0)-\psi(z,0)|\right)$$

and we define for $\kappa>0$

$$c_1^\kappa=\sup_{\{(x_1,x_2)\in\partial\Omega,\ x_1\neq x_2\}}\frac{(\omega_\psi^x(x_1-x_2)-\kappa)^+}{W_{x_1}(x_2)}$$

and

$$c_2^\kappa=\sup_{\{(t_1,t_2)\in[0,T],t_1\neq t_2\}}\frac{(\omega_\psi^t(t_1-t_2)-\kappa)^+}{|t_1-t_2|}$$

$$c_3^\kappa = \sup_{\{(x_1,x_2)\in\overline{\Omega}, x_1\neq x_2\}} \frac{(\omega_\psi^x(x_1-x_2)-\kappa)^+}{|x_1-x_2|^{q_1}}$$

Then the barriers are given by

$$W_1(x,t) = \inf_{\kappa\in\mathbb{R}^+, z\in\partial\Omega, \tau\in[0,T]} \{\psi(z,\tau)+2\kappa+\left(c_1^\kappa + (|f|_\infty + c_2^\kappa)^{\frac{1}{1+\alpha}}\right) W_z(x)+c_2^\kappa|t-\tau|\},$$

$$W_2(x,t) = \inf_{\kappa\in\mathbb{R}^+, y\in\overline{\Omega}} \{\psi(y,0)+2\kappa + c_3^\kappa|x-y|^{q_1} + \left(K_2(c_3^\kappa)^{1+\alpha} + |f|_\infty + c_2^\kappa\right) t\}$$

and $W = \inf(W_1, W_2)$. Similarly we define the sub-solution V.
This ends the proof of proposition 2.

Moreover by the comparison principle in theorem 1

$$V \leq W.$$

4 Existence and regularity.

In this section we assume as in section 3 that Ω is some bounded domain which satisfies the uniform exterior cone condition, that F satisfies (H1), (H2), (H3), and h satisfies (H5). We assume also that f and ψ are continuous and bounded.

For the regularity results presented later, we shall assume in addition that ψ is Holderian in x, lipschitz in time and f is Holderian in t.

We first prove, via Perron's method and with the aid of the sub and super-solutions just defined, that there exists a unique continuous solution u of

$$\begin{cases} u_t - F(x,\nabla u, D^2u) - h(x,t)\cdot\nabla u|\nabla u|^\alpha = f & \text{in } Q_T \\ u = \psi(x,t) & \text{on } \partial Q_T. \end{cases}$$

Next we prove some Hölder's estimates on this solution.

We consider V and W as in section 3 before, $V \leq W$, V is a sub-solution, and W is a super-solution. Let

$$E = \{u, \text{sub}-\text{solution of } 1_{\{f,\psi\}}\ ,\ V \leq u \leq W\}.$$

We need to prove that if $\bar{u} =: \sup E$, the lower semi-continuous enveloppe $\bar{u}_\star$ is a super solution of $1_{\{f,\psi\}}$, while $\bar{u}^\star$ is a sub-solution. This can be done using the two following propositions :

Proposition 3 *Suppose that Ω is some open set in $\mathbb{R}^N$. Suppose that u_n is some locally uniformly bounded sequence of sub-solutions for*

$$(u_n)_t - F(x, \nabla u_n, D^2 u_n) - h(x,t) \cdot \nabla u_n |\nabla u_n|^\alpha \leq f.$$

Let $\bar{u}$ be defined as

$$\bar{u}(\bar{x}, \bar{t}) = \lim_{r \to 0} \sup \{u_n(y,s), n \geq \frac{1}{r}, |t-s| + |y-x| \leq r\}$$

Suppose that f is upper semicontinuous. Then $\bar{u}$ is a sub-solution .

Proposition 4 *Suppose that f is continuous and V and W are respectively sub-solution and a super-solution with $V \leq W$, and let $\bar{u} = \sup\{\text{sub} - \text{solutions} \leq W\}$. Then $\bar{u}$ is a super-solution, hence it is a solution.*

Proof of proposition 3

$\bar{u}$ is upper semicontinuous by construction.

We assume that we are in the "bad " case, ie that $(\bar{x}, \bar{t})$ is such that there exists $\varphi \in \mathcal{C}^1$ which depends only on t, and there exists some δ_1, $\sup_{|t-\bar{t}| \leq \delta_1} (\bar{u}(x,t) - \varphi(t)) = \bar{u}(\bar{x}, \bar{t}) - \varphi(\bar{x}, \bar{t})$, with for some δ, $x \mapsto \sup_{|t-\bar{t}| \leq \delta_1} (\bar{u}(x,t) - \varphi(t))$ is constant on $B(\bar{x}, \delta)$. Then $\max_{x \in B(\bar{x},\delta), |t-\bar{t}| \leq \delta_1} (\bar{u}(x,t) - \varphi(t)) = \bar{u}(\bar{x}, \bar{t}) - \varphi(\bar{x}, \bar{t})$.

Let $k > \sup(2, \frac{\alpha+2}{\alpha+1})$.

We also have $\sup_{x \in B(\bar{x},\delta), |t-\bar{t}| < \delta_1} \{\bar{u}(x,t) - \varphi(t)) - |x-\bar{x}|^k - |t-\bar{t}|^2\} = \bar{u}(\bar{x}, \bar{t}) - \varphi(\bar{x}, \bar{t})$ and the supremum is strict in x and t separately.

We now consider

$$\sup_{x \in B(\bar{x},\delta), |t-\bar{t}| < \delta_1} \{u_n^\star(x,t) - \varphi(t) - |x-\bar{x}|^k - |t-\bar{t}|^2\}$$

This supremum is achieved on some (x_n, t_n). We begin to observe that $u_n^\star(x_n, t_n) \to \bar{u}(\bar{x}, \bar{t})$. Indeed by definition of $\bar{u}$, there exists (y_n, s_n) which goes to $(\bar{x}, \bar{t})$ and $u_n^\star(y_n, s_n) \to \bar{u}(\bar{x}, \bar{t})$. Then $u_n^\star(x_n, t_n) - \varphi(t_n) - |x_n - \bar{x}|^k - |t_n - \bar{t}|^2 \geq u_n^\star(y_n, s_n) - \varphi(t_n) - |y_n - \bar{x}|^k - |s_n - \bar{t}|^2 \to \bar{u}(\bar{x}, \bar{t})$, which implies that $\liminf u_n^\star(x_n, t_n) \geq \bar{u}(\bar{x}, \bar{t})$. On the other hand, using the definition of $\bar{u}$

$$\limsup_n u_n^\star(x_n, t_n) \leq \bar{u}(\bar{x}, \bar{t}).$$

Moreover since the supremum is strict, $(x_n, t_n) \to (\bar{x}, \bar{t})$.

If $\bar x \neq x_n$ for an infinity of n, using the fact that $(x,t)\mapsto \varphi(t)+|x-\bar x|^k+|t-\bar t|^2$ is a test function for $u_n^\star$ on (x_n,t_n) with a non zero gradient with respect to x on (x_n,t_n), one gets that for some constant C

$$\begin{aligned}\varphi'(t_n)+2(t_n-\bar t) &- Ck^{2+\alpha}|x_n-\bar x|^{k(\alpha+1)-\alpha-2} - k^{1+\alpha}|h|_\infty |x_n-\bar x|^{(k-1)(\alpha+1)}\\ &\leq \varphi'(t_n)+2(t_n-\bar t) - F(k|x_n-\bar x|^{k-2}(x_n-\bar x), D^2(|x-\bar x|^k)(x_n))\\ &- h(x_n,t_n)\cdot(x_n-\bar x)k^{1+\alpha}|x_n-\bar x|^{(k-1)(\alpha+1)-1}\\ &\leq f(x_n,t_n)\end{aligned}$$

This gives the result by passing to the limit since $k>\frac{\alpha+2}{\alpha+1}$ and f is upper semicontinuous. We now suppose that $x_n=\bar x$ for all n large enough. Then using lemma 1 in its form for sub- solutions one gets that

$$\varphi'(t_n)+2(t_n-\bar t)-0\leq f(\bar x,t_n).$$

Once more by passing to the limit and using the upper semi continuity of f we get the desired result.

When we are not in the "bad case", one can argue as in [14] and [4], Proposition 5.2, so we finally get that $\bar u$ is a sub-solution.

Proof of proposition 4

We suppose by contradiction that there exists some point $(\bar x,\bar t)$, some $\epsilon>0$ and some $\varphi\in\mathcal{C}^{1,2}$ such that $\nabla\varphi(\bar x,\bar t))\neq 0$ and

$$\varphi_t(\bar x,\bar t)-h(\bar x,\bar t)\cdot\nabla\varphi(\bar x,\bar t)|\nabla\varphi(\bar x,\bar t)|^\alpha - F(\bar x,\nabla\varphi(\bar x,\bar t),D^2\varphi(\bar x,\bar t))\leq f(\bar x,\bar t)-\epsilon$$

and $(\bar u_\star-\varphi)\geq(\bar u-\varphi)(\bar x,\bar t)=0$ on $B(\bar x,\delta_1)\times|t-\bar t|<\delta_1$, for some $\delta_1>0$.

We prove first that $\bar u_\star(\bar x,\bar t)<W(\bar x,\bar t)$. If not φ achieves also W by below on $(\bar x,\bar t)$ and since W is a super-solution one has

$$\varphi_t(\bar x,\bar t)-h(\bar x,\bar t)\cdot\nabla\tilde\varphi(\bar x,\bar t)|\nabla\tilde\varphi(\bar x,\bar t)|^\alpha - F(\bar x,\nabla\varphi(\bar x,\bar t),\ D^2\varphi(\bar x,\bar t))\geq f(\bar x,\bar t)$$

a contradiction. We have then $\varphi(\bar x,\bar t)<W(\bar x,\bar t)$.

Let $\tilde\varphi$ be defined as $\tilde\varphi(x,t)=\varphi(x,t)-|x-\bar x|^k-|t-\bar t|^k$ which achieves also $\bar u_\star$ by below on a neighborhood of $(\bar x,\bar t)$. We now choose $\delta<\delta_1$ small enough in order that for all $(x,t)\in\{|x-\bar x|\leq\delta\}\times\{|t-\bar t|\leq\delta\}$ one has $(W-\tilde\varphi)>0$ and by the continuity of F, h , $\tilde\varphi$, $\nabla\tilde\varphi$, $D^2\tilde\varphi$,

$$\begin{aligned}|\tilde\varphi_t(\bar x,\bar t)-\tilde\varphi_t(y,t)| &+ |h(\bar x,\bar t)\cdot\nabla\tilde\varphi(\bar x,\bar t)|\nabla\tilde\varphi(\bar x,\bar t)|^\alpha - h(y,t)\cdot\nabla\tilde\varphi(y,t)|\nabla\tilde\varphi(y,t)|^\alpha|\\ &+ |F(y,\nabla\tilde\varphi(y,t),D^2\tilde\varphi((y,t))-F(\bar x,\nabla\tilde\varphi(\bar x,\bar t),D^2\tilde\varphi(\bar x,\bar t))|\\ &+ |f(\bar x,\bar t)-f(y,t)|\leq\frac{\epsilon}{2}\end{aligned}$$

Let $0 < r < \inf_{\{|x-\bar{x}|\leq\delta\}\times\{|t-\bar{t}|\leq\delta\}}(W-\tilde{\varphi})$, and $r < \frac{\delta^k}{2^{k-1}}$. We now define $w = \sup(\tilde{\varphi}+r, \bar{u}_\star)$ in $B(\bar{x},\delta_1)\times\{|t-\bar{t}|<\delta_1\}$ and $w = \bar{u}_\star$ elsewhere. Then $w(\bar{x},\bar{t}) = \varphi(\bar{x},\bar{t})+r$ and in $|x-\bar{x}|+|t-\bar{t}| > \delta$, $w = \bar{u}_\star$. w is then a sub-solution which is greater than $\bar{u}_\star$ somewhere. We have reached a contradiction.

We now assume that there exists some $(\bar{x},\bar{t})$, some $\epsilon > 0$, some δ_1 such that for $|x-\bar{x}| < \delta_1$, and for some φ in $\mathcal{C}^1(\bar{t}-\delta_1, \bar{t}+\delta_1)$

$$\inf_{\{t,\ |t-\bar{t}|<\delta_1\}}(\bar{u}_\star - \varphi(t)) = \bar{u}_\star(\bar{x},\bar{t}) - \varphi(\bar{x},\bar{t}) = 0$$

and

$$\varphi_t(\bar{t}) \leq f(\bar{x},\bar{t}) - \epsilon.$$

We prove that $\bar{u}_\star(\bar{x},\bar{t}) = \varphi(\bar{x},\bar{t}) < W(\bar{x},\bar{t})$. Indeed, if not one has

$$\inf_{x\in B(\bar{x},\delta) t\in]\bar{t}-\delta_1,\bar{t}+\delta_1[}(W(x,t)-\varphi(t)) = W(\bar{x},\bar{t}) - \varphi(\bar{t}),$$

and using lemma 1 one would obtain since W is a super-solution that

$$\varphi'(\bar{t}) \leq f(\bar{x},\bar{t}),$$

a contradiction.

We now replace φ by $\tilde{\varphi}(x,t) = \varphi(t) - |x-\bar{x}|^k - |t-\bar{t}|^k$. We choose $\delta_2 < \delta_1$ small enough in order that $(W-\tilde{\varphi}) > 0$ on $B(\bar{x},\delta_2)\times\{|t-\bar{t}| < \delta_2\}$. Then, using the continuity of F, f, $\nabla\tilde{\varphi}$, $D^2\tilde{\varphi}$ one can choose $\delta < \delta_2$ small enough in order that for $|x-\bar{x}| \leq \delta$ and $|t-\bar{t}| \leq \delta$,

$$\tilde{\varphi}_t(x,t) - h(\bar{x},\bar{t})\cdot\nabla\tilde{\varphi}(x,t)|\nabla\tilde{\varphi}(x,t)|^\alpha - F(x,\nabla\tilde{\varphi}(x,t), D^2\tilde{\varphi}(x,t)) \leq f(x,t) - \frac{\epsilon}{2}$$

From this and using lemma 1 for the point $\bar{x}$ one sees that on $|x-\bar{x}| < \delta$ and $|t-\bar{t}| \leq \delta$ $(x,t) \mapsto \tilde{\varphi}(x,t)$ is a sub-solution.

We choose $r < \inf_{|x-\bar{x}|\leq\delta_2,\ |t-\bar{t}|<\delta_2}(W-\tilde{\varphi})(x,t)$ and such that $r < \frac{\delta^k}{2^{k-1}}$. We then define $w = \sup(\tilde{\varphi}(x,t)+r, \bar{u}_\star)$ in the set $\{|x-\bar{x}| < \delta_2, |t-\bar{t}| < \delta_2\}$, then w is a sub-solution which coincide with $\bar{u}_\star(\bar{x},\bar{t})+r$ on $(\bar{x},\bar{t})$ and with $\bar{u}_\star$ for $|x-\bar{x}|+|t-\bar{t}| > \delta$. We have obtained a sub-solution which is less than W and strictly greater than $v_\star$ on some open set. This contradicts the definition of $\bar{u}$ and then $\bar{u}_\star$ is a super-solution. This ends the proof of proposition 3.

By the comparison principle in Theorem 1, we get that $\bar{u}_\star \geq \bar{u}^\star$ hence the function $\bar{u}$ is continuous and it is the required solution. We have obtained the following existence's result :

Theorem 2 *Suppose that f and ψ are continuous and bounded on Q_T. Then there exists a unique continuous solution of $1_{\{f,\psi\}}$.*

Remark 5 *The uniqueness is given by the comparison principle.*

We now prove some Hölder's estimate :

Theorem 3 *Suppose that f is continuous, bounded on Q_T, and Hölder's continuous of exponent γ_f with respect to t, that ψ is Hölder's continuous with exponent γ with respect to x and Lipschitzian in t, and let u be the solution of $1_{\{f,\psi\}}$. Then there exists some constant c, such that for all $(x,t),(y,s)$ in Q_T^2, and for $q = \frac{q_1}{\gamma} = \sup\left(\frac{\alpha+2}{\gamma(\alpha+1)}, \frac{2}{\gamma}\right)$ $\gamma^\star = \inf(\gamma_f, \frac{1}{q(\alpha+1)-\alpha})$*

$$|u(x,t) - u(y,s)| \leq c(|x-y|^\gamma + |t-s|^{\gamma^\star}).$$

Corollary 1 *Suppose that (f_n) is a sequence of uniformly bounded functions, continuous w.r.t. x and uniformly Hölderian in t, and (ψ_n) is uniformly Hölder's continuous in x and uniformly Lipshitzian in t, then the sequence (u_n) of solutions of $1_{\{f_n,\psi_n\}}$ is uniformly Hölder's continuous and bounded.*

In order to prove Theorem 3 we give three preliminary results, which establish some Hölder's estimates on the bottom and on the lateral boundary of Q_T.

Proposition 5 *Let $Q_T = \Omega\times]0,T[$.*

Let ψ be an Hölder function with exponent γ in x and Lipshitzian in t on ∂Q_T, let f be continuous on $\overline{Q_T}$ and let u be the solution of

$$\begin{cases} \partial_t u = F(x, \nabla u, D^2 u) + h(x,t)\cdot\nabla u|\nabla u|^\alpha + f(x,t) & \text{in } Q_T \\ u(x,t) = \psi(x,t) & \text{on } (\partial\Omega\times[0,T]) \cup (\overline{\Omega}\times\{0\}) \end{cases}$$

Then there exists some constant C_2 such that, for all $(x,t)\in\Omega\times]0,T[$,

$$|u(x,t) - \psi(x,0)| \leq C_2 t^{\frac{1}{q(\alpha+1)-\alpha}}$$

(We recall that $q = \frac{\sup(2,\frac{\alpha+2}{\alpha+1})}{\gamma}$).

Proof.

By the comparison principle in theorem 1 one has

$$\begin{aligned} u(x,t) &\leq W(x,t) \\ &\leq W_2(x,t) \\ &\leq \psi(x,0) + \inf_{\kappa\in\mathbb{R}^+}\left(\kappa + \left(\frac{c_\psi^q}{c_q^q \kappa^{q-1}}\right)^{1+\alpha} K_2 t\right) + (|f|_\infty + |\psi_t|_\infty)t \\ &= \psi(x,0) + Ct^{\frac{1}{(q-1)(1+\alpha)+1}} + (|f|_\infty + |\psi_t|_\infty)t \end{aligned}$$

for some constant C which depends on $(c_\psi, A, a, q_1, \gamma)$, computed with the aid of (3.4), replacing q by $(q-1)(\alpha+1)+1$.

This yields the result. The symmetric lower bound is obtained by considering V instead of W and proceeding similarly.

As a consequence one has the following

Proposition 6 *We assume here that f is continuous on $\overline{Q_T}$, Hölder with respect to t, for some exponent γ_f. Let u be a solution of $1_{\{f,\psi\}}$. Then there exists C_2 depending on the Hölder's constant c_ψ and c_f of ψ and f respectively , such that for all $x\in\Omega$ and for all $(t,s)\in]0,T[^2$,*

$$|u(x,t+s) - u(x,t)| \leq C_2 s^{\gamma^\star}.$$

where $\gamma^\star = \inf(\frac{1}{q(\alpha+1)-\alpha}, \gamma_f)$, $q = \frac{\sup(2,\frac{\alpha+2}{\alpha+1})}{\gamma} = \frac{q_1}{\gamma}$.

Proof of Proposition 6: Let c_f be such that for all $x\in\Omega$ and for all $(t,s)\in]0,T[^2$,

$$|f(x,t+s) - f(x,t)| \leq c_f s^{\gamma_f}.$$

We define for s fixed in]0, T[

$$\begin{aligned} v(x,t) &= u(x,t+s) + t\ c_f s^{\gamma_f} + \sup_{(x,t)\in\partial\Omega\times]0,T-s[} |\psi(x,t+s) + c_f t s^{\gamma_f} - \psi(x,t)| \\ &+ \sup_{x\in\Omega} |u(x,s) - \psi(x,0)| \end{aligned}$$

Then v satisfies on $\Omega\times]0,T-s[$

$$\partial_t v - F(x,\nabla v, D^2 v) - h(x,t)\cdot\nabla v|\nabla v|^\alpha = f(x,t+s) + c_f s^{\gamma_f} \geq f(x,t)$$

Since u satisfies the opposite inequality on the same open set, and by construction $v(x,t) \geq u(x,t)$ on ∂Q_T, one has by theorem 1

$$u(x,t) - v(x,t) \leq 0,$$

which gives the result, redefining $C_2 = 2T^{1+\gamma_f-\gamma^\star} + |\psi_t|_\infty T^{1-\gamma^\star} + C_2 T^{\frac{1}{q(\alpha+1)-\alpha}-\gamma^\star}$
For the reverse inequality, one uses for s fixed

$$\begin{aligned} v(x,t) &= u(x,t+s) - t\, c_f s^{\gamma_f} - \sup_{(x,t)\in\partial\Omega\times]0,T-s[} |\psi(x,t+s) + c_f t s^{\gamma_f} - \psi(x,t)| \\ &- \sup_{x\in\Omega} |u(x,s) - \psi(x,0)| \end{aligned}$$

v is a sub-solution of

$$v_t - F(x, \nabla v, D^2 v) - h(x,t)\cdot \nabla v |\nabla v|^\alpha \leq f(x,t+s) - c_f s^{\gamma_f} \leq f(x,t)$$

and $u(x,t)$ satisfies the opposite inequality on $]0, T-s[$. Moreover $v(x,t) \leq u(x,t)$ on ∂Q_T. Then Theorem 1 implies that

$$u(x,t+s) \leq u(x,t) + C_2 s^{\gamma^\star}$$

with C_2 as above.

We now give an estimate on the lateral boundary :

Proposition 7 *We assume that ψ is Hölder continuous of exponent γ with respect to x and Lipschitzian with respect to t. Let u be a solution of $1_{\{f,\psi\}}$. Then there exists C_1 such that for all $(x, x_o) \in \Omega \times \partial\Omega$ and $t \in [0,T)$,*

$$|u(x,t) - u(x_o,t)| \leq C_1 |x - x_o|^\gamma.$$

Proof

We use once more the super-solution. Taking in the infimum defining W the point (x_o, t) which is on the lateral boundary, and using the properties of the barrier, one has

$$\begin{aligned} u(x,t) &\leq W(x,t) \\ &\leq W_1(x,t) \\ &\leq \psi(x_o,t) + \frac{c_\psi}{\underline{c}} W_{x_o}(x) + (|f|_\infty + |\psi_t|_\infty)^{\frac{1}{1+\alpha}} W_{x_o}(x) \\ &\leq \psi(x_o,t) + \left(\frac{c_\psi}{\underline{c}} + (|f|_\infty + |\psi_t|_\infty)^{\frac{1}{1+\alpha}}\right)\bar{c}|x - x_o|^\gamma. \end{aligned}$$

This gives the result with

$$C_1 = \bar{c}(\frac{c_\psi}{\underline{c}} + (|f|_\infty + |\psi_t|_\infty)^{\frac{1}{1+\alpha}})$$

One gets the lower bound by considering V instead of W.

We now prove Theorem 3. First observe that u is bounded as soon as f and ψ are bounded, due to theorem 1, the inequalities $V \leq u \leq W$, and the definition of V and W.

In the following δ will be $< \inf(1, \frac{1}{T})$, and $L > 1$.

We construct a function Φ as follows: Let δ be small enough in order that, for $\tilde{\omega}$ the modulus of continuity given in the assumption (H3), and C being the universal constant defined in (4.7) later, one has $\tilde{\omega}(\delta) < \frac{a}{4C}$, and $\delta |h|_\infty < \frac{a}{C}$. We define

$$L = \sup \left(C_1, \left(\frac{2|f|_\infty \delta^{\alpha+1-(\alpha+2)\gamma}}{a\,(\gamma)^{1+\alpha}\,(1-\gamma)} \right)^{\frac{1}{1+\alpha}}, \frac{2 \sup u}{\delta^\gamma} \right)$$

$$M = \sup(C_2, \frac{2 \sup u}{\delta^{\gamma^\star}})$$

where C_1 is given in Proposition 7, and C_2 is given in Proposition 6. We also define

$$\Delta_\delta = \{((x,t),(y,s)) \in Q_T^2,\ |x-y| < \delta, |t-s| < \delta\}.$$

Claim *For any* $(x,t),(y,s) \in \Delta_\delta$

$$\Phi(x,t,y,s) = u(x,t) - u(y,s) - L|x-y|^\gamma - M|t-s|^{\gamma^\star} \leq 0. \tag{4.5}$$

We argue by contradiction and suppose that the supremum of ϕ is positive. Then, for κ small enough > 0 the supremum of $\phi - \frac{\kappa}{T-t} - \frac{\kappa}{T-s}$ is also strictly positive. In the following we replace ϕ by $\phi - \frac{\kappa}{T-t} - \frac{\kappa}{T-s}$.

From the choice of the constants and Propositions 6 and 7 we know that the inequality (4.5) with the "new " ϕ holds on $\partial\Delta_\delta$:

Indeed if $x \in \partial\Omega$, $y \in \Omega$, and $(t,s) \in]0,T[^2$, $|s-t| < \delta$, using Proposition 7, one has

$$\begin{array}{rcl} u(x,t) - u(y,s) & \leq & \psi(x,t) - \psi(x,s) + u(x,s) - u(y,s) \\ & \leq & |\psi_t|_\infty |t-s| + C_1|x-y|^\gamma \end{array}$$

which gives the result since $M \geq C_2 \geq |\psi_t|_\infty T^{1-\gamma^\star}$ and $L \geq C_1$. The same is true by exchanging x and y.

If $|x-y|=\delta$ or $|t-s|=\delta$, the result holds by the choice of L and M. For $t=0$ or $s=0$, one uses proposition 6 and proposition 7 to get $|u(x,t)-u(y,0)| \le |u(x,t)-u(x,0)| + |u(x,0)-u(y,0)| \le c_\psi |x-y|^\gamma + C_2 t^{\gamma^\star}$, from which we conclude since $L > c_\psi$ and $M > C_2$.

Finally the supremum cannot be achieved on $t=T$ or $s=T$ since in that case the function equals $-\infty$.

Suppose by contradiction that

$$\sup_{(x,t),(y,s)\in Q_T^2} \Phi(x,t,y,s) > 0.$$

Then for $n>0$ large enough

$$\Phi_n(x,t,y,s) = u(x,t) - u(y,s) - L|x-y|^\gamma - M(|t-s|^2 + n^{-2})^{\frac{\gamma^\star}{2}} - \frac{\kappa}{T-t} - \frac{\kappa}{T-s}$$

has also a supremum >0, and it cannot be achieved on the boundary, by the previous considerations. We denote by $(\bar{x}_n, \bar{t}_n), (\bar{y}_n, \bar{s}_n)$ a couple inside Δ_δ on which the supremum of ψ_n is achieved, and in the following we fix n large enough and drop the indexes n for simplicity.

Suppose that $\bar{x} = \bar{y}$. Then one would have

$$u(\bar{x},\bar{t}) - u(\bar{x},\bar{s}) \ge M((\bar{t}-\bar{s})^2 + \frac{1}{n^2})^{\frac{\gamma^\star}{2}},$$

which contradicts proposition 6 and the choice of M. Hence $\bar{x} \neq \bar{y}$ and using Ishiis' lemma (see also lemma 2.1 in [3]), there exists $X \in S$ and Y in S such that:

$$\left(M\gamma^\star \frac{\bar{t}-\bar{s}}{((\bar{t}-\bar{s})^2 + \frac{1}{n^2})^{1-\frac{\gamma^\star}{2}}} + \frac{\kappa}{(T-\bar{t})^2}, \gamma L(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{\gamma-2}, X \right) \in \bar{J}^{2,+}u(\bar{x},\bar{t})$$

$$\left(M\gamma^\star \frac{\bar{t}-\bar{s}}{((\bar{t}-\bar{s})^2 + \frac{1}{n^2})^{1-\frac{\gamma^\star}{2}}} - \frac{\kappa}{(T-\bar{s})^2}, \gamma L(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{\gamma-2}, -Y \right) \in \bar{J}^{2,-}u(\bar{y},\bar{s})$$

with

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and $B = L\gamma|\bar{x}-\bar{y}|^{\gamma-2}(I + (\gamma-2)\frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2})$.

We need a more precise estimate, as in [15]. For that aim let P the symmetric matrix defined as :

$$0 \leq P := \frac{(\bar{x}-\bar{y}\otimes\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2} \leq I.$$

Using $-(X+Y)\geq 0$, $(I-P)\geq 0$ and the properties of the symmetric matrices one has

$$tr(X+Y) \leq tr(P(X+Y)).$$

Remarking in addition that $X+Y\leq 4B$, one sees that $tr(X+Y)\leq tr(P(X+Y))\leq 4tr(PB)$. But $tr(PB)=\gamma L(\gamma-1)|\bar{x}-\bar{y}|^{\gamma-2}<0$, hence

$$|tr(X+Y)| \geq 4\gamma L(1-\gamma)|\bar{x}-\bar{y}|^{\gamma-2}. \tag{4.6}$$

Furthermore by Lemma III.1 of [15] there exists a universal constant C such that

$$|X|,|Y| \leq C(|tr(X+Y)|+|B|^{\frac{1}{2}}|tr(X+Y)|^{\frac{1}{2}}) \leq C|tr(X+Y)|, \tag{4.7}$$

since $|B|$ and $|tr(X+Y)|$ are of the same order. This constant is the constant used for the choice of L at the beginning of the proof.

Using the fact that u is both a sub- and a super-solution we get

$$\begin{aligned}
f(\bar{x},\bar{t}) &\geq M\gamma^\star\left(\frac{\bar{t}-\bar{s}}{((\bar{t}-\bar{s})^2+\frac{1}{n^2})^{1-\frac{\gamma^\star}{2}}}\right)+\frac{\kappa}{(T-\bar{t})^2}\\
&- F(\bar{x},\gamma L(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{\gamma-2},X)\\
&- L^{1+\alpha}\gamma^{1+\alpha}h(\bar{x},\bar{t})\cdot(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{(\gamma-1)(\alpha+1)-1}\\
&\geq M\gamma^\star\left(\frac{\bar{t}-\bar{s}}{((\bar{t}-\bar{s})^2+\frac{1}{n^2})^{1-\frac{\gamma^\star}{2}}}\right)-\frac{\kappa}{(T-\bar{s})^2}-F(\bar{y},\gamma L(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{\gamma-2},-Y)\\
&- L^{1+\alpha}\gamma^{1+\alpha}h(\bar{y},\bar{s})\cdot(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{(\gamma-1)(\alpha+1)-1}-\tilde{\omega}(|\bar{x}-\bar{y}|)(\gamma L|\bar{x}-\bar{y}|^{\gamma-1})^\alpha|X|\\
&- L^{1+\alpha}|h|_\infty\gamma^{1+\alpha}|\bar{x}-\bar{y}|^{(\gamma-1)(\alpha+1)}+(\gamma L|\bar{x}-\bar{y}|^{\gamma-1})^\alpha a|tr(X+Y)|\\
&\geq f(\bar{y},\bar{s})+4\gamma^{1+\alpha}L^{1+\alpha}(1-\gamma)|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)(\alpha+1)}(a-\frac{\tilde{\omega}}{C}(|\bar{x}-\bar{y}|)-\frac{|h|_\infty}{2C}|\bar{x}-\bar{y}|)
\end{aligned}$$

which is a contradiction with the assumptions on L. We have obtained that

$$u(x,t)-u(y,s) \leq L|x-y|^\gamma + M\frac{|t-s|^{\gamma^\star}}{T-t}.$$

This ends the proof.

5 Global solutions on $\Omega \times \mathbb{R}^+$

In this section we prove the existence of solutions on $\Omega \times \mathbb{R}^+$. For this we establish some two sided property of solutions around $t = T$.

Proposition 8 *We suppose that f is continuous and bounded on $\Omega \times \mathbb{R}^+$, and that u is a super-solution of $1_{\{f,\psi\}}$ on Q_T, lower semicontinuous. We define*

$$u(x,T) = \liminf_{|z-x|+|t-T|\le r} u(z,t).$$

Then u being extended in that kind is a super-solution on $\Omega\times]0,T]$.

In the same manner if v is a upper semicontinuous sub-solution, we define

$$v(x,T) = \limsup_{|z-x|+|t-T|\le r} v(z,t).$$

Then v being extended in that kind is a sub-solution on $\Omega\times]0,T]$.

Proof

We follow partly the arguments employed in [20].

Let u be a super-solution and let φ be a $\mathcal{C}^{1,2}$ function such that

$$(u-\varphi)(x,t) \ge (u-\varphi)(\bar{x},T)$$

for (x,t) on some neighborhood V of $(\bar{x},T)$, and $\nabla\varphi(\bar{x},\bar{t}) \neq 0$. One can assume replacing if necessary $\varphi(x,t)$ by $\varphi(x,t) - |x-\bar{x}|^k - |t-T|^2$ for $k > \sup(2, \frac{\alpha+2}{\alpha+1})$, that the infimum of $(u-\varphi)$ is strict on $(\bar{x},T)$.

Then for n large enough

$$\inf_{(x,t)\in V} \left(u(x,t) - \varphi(x,t) + \frac{1}{n(T-t)} \right)$$

is achieved on (y_n,t_n) with $(y_n,t_n) \to (\bar{x},T)$.

Indeed we prove first that

$$\lim_{n\to+\infty} \inf_{(x,t)\in V} \left(u(x,t) - \varphi(x,t) + \frac{1}{n(T-t)} \right) = \inf_{(x,t)\in V} (u-\varphi)(x,t).$$

Indeed, we already have

$$\inf_{(x,t)\in V}\left(u(x,t)-\varphi(x,t)+\frac{1}{n(T-t)}\right)\geq \inf(u-\varphi)(x,t).$$

For the reverse inequality let ϵ be given and (x_ϵ,t_ϵ) in Q_T with

$$(u-\varphi)(x_\epsilon,t_\epsilon)\leq \inf_{(x,t)\in V}(u-\varphi)(x,t)+\epsilon$$

then for $n(T-t_\epsilon)>\frac{1}{\epsilon}$

$$(u-\varphi)(x_\epsilon,t_\epsilon)+\frac{1}{n(T-t_\epsilon)}\leq (u-\varphi)(x_\epsilon,t_\epsilon)+2\epsilon\leq \inf_{(x,t)\in V}(u-\varphi)+2\epsilon.$$

ϵ being arbitrary, one gets the result.

Now the function $u-\varphi+\frac{1}{n(T-t)}$ being lower semi-continuous the infimum is achieved on some (y_n,t_n). By the previous considerations

$$\inf_{(x,t)\in V}(u-\varphi)(x,t)\leq (u-\varphi)(y_n,t_n)+\frac{1}{n(T-t_n)}\to (u-\varphi)(\bar{x},T).$$

This implies in particular that

$$(u-\varphi)(y_n,t_n)\to (u-\varphi)(\bar{x},T)$$

and since the infimum of $u-\varphi$ is strict, $(y_n,t_n)\to(\bar{x},T)$. Let us note that t_n does not go to T too much quickly, since $n(T-t_n)\to+\infty$.

Let $\varphi_n=\varphi(x,t)-\frac{1}{n(T-t)}$, since φ is $\mathcal{C}^1$, for n large enough, $\nabla\varphi_n(y_n,t_n)\neq 0$, and since φ_n achieves u by below on (y_n,t_n),

$$\frac{d}{dt}\varphi_n(y_n,t_n)-F(y_n,\nabla\varphi(y_n,t_n),D^2\varphi(y_n,t_n))-h(y_n,t_n)\cdot\nabla\varphi(y_n)|\nabla\varphi(y_n)|^\alpha\geq f(y_n,t_n),$$

hence

$$\begin{aligned}\frac{d}{dt}\varphi(y_n,t_n) &- F(y_n,\nabla\varphi(y_n,t_n),D^2\varphi(y_n,t_n))-h(y_n,t_n)\cdot\nabla\varphi(y_n)|\nabla\varphi(y_n)|^\alpha\\ &\geq f(y_n,t_n)+\frac{1}{n(T-t)^2}\\ &\geq f(y_n,t_n),\end{aligned}$$

and passing to the limit one gets that

$$\frac{d}{dt}\varphi(\bar{x},T) - F(\bar{x},\nabla\varphi(\bar{x},T),D^2\varphi(\bar{x},T)) - h(\bar{x},T)\cdot\nabla\varphi(\bar{x},T)|\nabla\varphi(\bar{x},T)|^\alpha \geq f(\bar{x},T).$$

This ends the case $\nabla\varphi(\bar{x},T)\neq 0$.

We now assume that there exists some $\mathcal{C}^1$ function φ which depends only on t, and some $\delta_1>0$ such that $u(\bar{x},T)-\varphi(T)=\inf_{|t-\bar{t}|<\delta_1}(u(x,t)-\varphi(t))$ and $\inf_{|t-\bar{t}|<\delta_1}\{u(x,t)-\varphi(t)\}$ is constant in a neighborhood $B(\bar{x},\delta)$ of $\bar{x}$. Then one also has

$$\inf_{x\in B(\bar{x},\delta),|t-\bar{t}|<\delta_1}\{u(x,t)-\varphi(t)+|x-\bar{x}|^k+|t-T|^2\} = u(\bar{x},T)-\varphi(T)$$

Defining $\varphi_n(t)=\varphi(t)-|x-\bar{x}|^k-|t-T|^2-\frac{1}{n(T-t)}$ one gets also that there exists (x_n,t_n) which converges to $(\bar{x},T)$ and (x_n,t_n) is a local minimum for $u-\varphi_n$.

- Either $x_n=\bar{x}$ for all n large enough, then using lemma 1 one gets

$$\partial_t\varphi(t_n)-2(t_n-T)-\frac{1}{n(T-t_n)^2}\geq f(\bar{x},t_n).$$

which yields the result by passing to the limit.

-Or for an infinity of n, $x_n\neq\bar{x}$, then

$$\begin{aligned}\partial_t\varphi(t_n)-2(t_n-T)-\frac{1}{n(T-t_n)^2} &- F(x_n,-k|x_n-\bar{x}|^{k-2}(x_n-\bar{x}),-D^2(|\bar{x}-x|^k)(x_n))\\ &+ k^{1+\alpha}h(x_n,t_n)\cdot(x_n-\bar{x})|x_n-\bar{x}|^{(k-1)(\alpha-1)-1}\\ &\geq f(x_n,t_n).\end{aligned}$$

Since $|\bar{x}-x_n|$ and $|t_n-T|$ tend to zero when n goes to infinity, and $k>\frac{\alpha+2}{\alpha+1}$, one gets by passing to the limit that

$$\varphi'(T)\geq f(\bar{x},T).$$

From this one gets the existence of global solution on $\Omega\times\mathbb{R}^+$.

6 The case $\mathbb{R}^N\times]0,T[$

In this section we still assume that F satisfies (H1), (H2) , (H3) on $\mathbb{R}^N\times\mathbb{R}^N\setminus\{0\}\times S$ and h satisfies (H5).

We consider here the case of $\mathbb{R}^N\times]0,T[$. We assume in addition that F satisfies the uniform Lipschitz condition :

$(H6)$ There exists some constant C such that for all $p\neq 0$, for all X and for all q, such that $|q|<\frac{|p|}{2}$, and for all $x\in\mathbb{R}^N$, one has

$$|F(x,p+q,X)-F(x,p,X)|\leq C|p|^{\alpha-1}|q||X|$$

We prove here the following existence theorem

Theorem 4 *We suppose that ψ is uniformly continuous and bounded on $\mathbb{R}^N$ and f is uniformly continuous and bounded on $\mathbb{R}^N\times[0,T]$. Then there exists a unique continuous and bounded viscosity solution of*

$$\begin{cases} u_t - F(x,\nabla u, D^2u) - h(x,t)\cdot\nabla u|\nabla u|^\alpha = f(x,t) & \text{in } \mathbb{R}^N\times]0,T[\\ u(x,0)=\psi(x) & \text{for all } x\in\mathbb{R}^N. \end{cases}$$

The proof of theorem 4 follows the lines of the previous sections, in the case where Ω is bounded. We then need to produce a super-solution and a sub-solution and prove some comparison principle.

The super-solution is obtained with the aid of the following proposition

Proposition 9 *There exists G, some positive $\mathcal{C}^2$ function on $[0,\infty[$, and some constant B such that $u(x)=G(|x|)$ satisfies on $\mathbb{R}^N\times]0,T[$*

$$F(x,\nabla u,D^2u)+h(x,t)\cdot\nabla u|\nabla u|^\alpha\leq B.$$

Proof :

Let $q_1=\sup(2,\frac{\alpha+2}{\alpha+1})$, and define

$$G(r)=\begin{cases} r^{q_1} & \text{if } r<1 \\ \frac{q_1(1+q_1)r}{2}+\frac{q_1(q_1-1)}{2r}+1-q_1^2 & \text{if } r>1. \end{cases}$$

With this choice of G by a tedious but straithforward computation there exists some constant B such that for $u(x)=G(|x|)$

$$F(x,\nabla u,D^2u)+h(x,t)\cdot\nabla u|\nabla u|^\alpha\leq B.$$

Proof of theorem 4

We give the proof in the case where ψ is Hölder's continuous and we shall give the changes to bring in the case where ψ is only continuous. We denote by γ_ψ and c_ψ respectively some Hölder's exponent and some Hölder's constant for ψ. Let $q = \frac{q_1}{\gamma_\psi}$.

We define on the model of W_2 in section 3,

$$\begin{aligned} W(x,t) = \inf_{y \in \mathbb{R}^N, \kappa \in \mathbb{R}^+} \{\psi(y) + \kappa \quad &+ \quad \frac{(c_\psi + 2|\psi|_\infty)^q}{c_q^q \kappa^{q-1}} G(|y-x|) + |f|_\infty t \\ &+ \quad \left(\frac{(c_\psi + 2|\psi|_\infty)^q}{c_q^q \kappa^{q-1}} \right)^{1+\alpha} Bt\}. \end{aligned}$$

Then W is an infimum of super-solutions for $1_{\{|f|_\infty\}}$. Moreover

$$W(x,0) = \inf_{\{|y-x|<1, \kappa \in \mathbb{R}^+\}} (\psi(y) + \kappa + \frac{(c_\psi + 2|\psi]_\infty)^q}{c_q^q \kappa^{q-1}} G(|x-y|) \geq \psi(y) + c_\psi |x-y|^{\gamma_\psi} \geq \psi(x),$$

and also using $G(r) \geq r$ for $r \geq 1$

$$\inf_{|y-x|>1} \{\psi(y) + (c_\psi + 2|\psi|_\infty)|y-x|^{\frac{1}{q}}\} \geq \psi(y) + 2|\psi|_\infty \geq \psi(x).$$

This implies that $W(x,0) \geq \psi(x)$. Moreover taking $y = x$ in the infimum, one gets

$$W(x,0) \leq \kappa + \psi(x),$$

for all κ. We have obtained that $W(x,0) = \psi(x)$. We now observe that W is uniformly bounded, indeed

$$\begin{aligned} -|\psi|_\infty \leq W(x,t) \quad &\leq \quad \inf_{\kappa \in \mathbb{R}^+} \{\psi(x) + \kappa + |f|_\infty t + \left(\frac{(c_\psi + 2|\psi|_\infty)^q}{c_q^q \kappa^{q-1}} \right)^{1+\alpha} Bt\} \\ &\leq \quad \psi(x) + 1 + |f|_\infty t + \left(\frac{(c_\psi + 2|\psi|_\infty)^q}{c_q^q} \right)^{1+\alpha} Bt \\ &\leq \quad |\psi|_\infty + C \end{aligned}$$

for some constant C which depends on the data and T.

Moreover

$$W(x,t) \leq \psi(y) + c_1(|x-y|^{\gamma_\psi}) + c_2 t^{\frac{1}{q(\alpha+1)-\alpha}} \tag{6.8}$$

Indeed taking $y = x$ in the infimum defining W and by the computation of the infimum with respect to κ one has

$$W(x,t) \leq \psi(x) + ct^{\frac{1}{q(\alpha+1)-\alpha}} \leq \psi(y) + c_\psi |x-y|^{\gamma_\psi} + ct^{\frac{1}{q(\alpha+1)-\alpha}}.$$

This will be used for the Hölder's estimates later.

Let us note that

$$V(x,t) = \sup_{y\in\mathbb{R}^N,\kappa\in\mathbb{R}^+} \{\psi(y,0) - \kappa - \frac{(c_\psi + 2|\psi|_\infty)^q}{c_q^q \kappa^{q-1}} G(|y-x|) - |f|_\infty t - \left(\frac{(c_\psi + 2|\psi|_\infty)^q}{c_q^q \kappa^{q-1}}\right)^{1+\alpha} Bt\}$$

with B as before , is a sub-solution of $1_{\{-|f|_\infty,\psi\}}$. Moreover V is bounded and satisfies for some constants c_1 and c_2

$$V(x,t) \geq \psi(y) - c_1|x-y|^{\gamma_\psi} - c_2 t^{\frac{1}{q(\alpha+1)-\alpha}}. \tag{6.9}$$

In the case where ψ is only continuous we introduce

$$c_3^\kappa = \sup_{\delta\in\mathbb{R}^+} \frac{(\omega(\delta)-\kappa)^+}{G(\delta)}$$

where ω the modulus of continuity of ψ and we define

$$W(x,t) = \inf_{y\in\mathbb{R}^N,\kappa\in\mathbb{R}^+} \{\psi(y) + \kappa + c_3^\kappa G(|x-y|) + |f|_\infty t + (c_3^\kappa)^{1+\alpha} Bt\}$$

and

$$V(x,t) = \sup_{y\in\mathbb{R}^N,\kappa\in\mathbb{R}^+} \{\psi(y) - \kappa - c_3^\kappa G(|x-y|) - |f|_\infty t - (c_3^\kappa)^{1+\alpha} Bt\}$$

V and W hence defined are uniformly bounded. Using theorem 5 below one gets that $V \leq W$. Then using Perron's method in section 4, whose proof does not use the boundedness of Ω, we obtain that there exists a solution of $1_{\{f,\psi\}}$ on $\mathbb{R}^N\times]0,T[$, in the sense that $u^\star$ is a sub-solution and $u_\star$ is a super-solution. We now use the fact that $V \leq u^\star$ and $u_\star \leq W$ to derive using theorem 5 that $u_\star \geq u^\star$, hence u is continuous. Applying once more theorem 5 one gets that the solution is unique. This ends the proof of theorem 4.

Theorem 5 *Suppose that f and g are uniformly continuous and bounded and $f \geq g$. Suppose that u and v are respectively upper semicontinuous and lower semicontinuous sub-and super-solutions of*

$$u_t - F(x,\nabla u, D^2u) - h(x,t)\cdot\nabla u|\nabla u|^\alpha \leq g(x,t) \text{ in } \mathbb{R}^N\times]0,T[$$

$$v_t - F(x, \nabla v, D^2 v) - b(x,t) \cdot \nabla v |\nabla v|^\alpha \geq f(x,t) \text{ in } \mathbb{R}^N \times]0, T[$$

with $u(x,0) \leq v(x,0)$*. Suppose in addition that there exists some constant* c_1 *such that for all* (x,y) *in* $\mathbb{R}^N$*,*

$$u(x,t) \leq u(y,0) + c_1(|x-y|+1) \tag{6.10}$$

and

$$v(x,t) \geq v(y,0) - c_1(|x-y|+1) \tag{6.11}$$

Then $u(x,t) \leq v(x,t)$ *for all* $(x,t) \in \mathbb{R}^N \times [0,T]$*.*

Proof of theorem 5

One can replace v by $(v)_\kappa = v + \frac{\kappa}{T-t}$. Then v_κ is a strict super-solution, which is infinite on $t = T$.

We shall prove that $u \leq v_\kappa$ and next we shall let κ go to zero. In the following we drop the index κ.

Suppose by contradiction that there exists $(\bar{x}, \bar{t})$ such that $(u-v)(\bar{x}, \bar{t}) > 0$. Then $\bar{t} < T$ according to the previous property of v.

We introduce for $j \in \mathbf{N}$ and for $k = \sup(3, \frac{|\alpha|}{3}, \frac{\alpha+2}{\alpha+1}, \alpha+1, \frac{\alpha+2}{6}, \frac{2(1+\alpha)}{\omega_h})$, the function ψ_j defined as

$$\psi_j(x,y,t,s) = u(x,t) - v(y,s) - \frac{j|x-y|^k}{k} - \frac{1}{j^{3k}}|x|^k - \frac{j}{2}|t-s|^2.$$

Then for j large enough the supremum of ψ_j is still > 0, for example as soon as

$$j^{3k} > \frac{|\bar{x}|^k}{u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t})}$$

In the following C will denote some constant which can vary from one line to another.

We prove first that if $\psi_j(x_j, y_j, t_j, s_j) > 0$, $j|x_j - y_j|^k \leq C$. Indeed, using the fact that $u(x,t) - v(y,s) \leq 2c_1(|x-y|+1)$, one gets that $j|x_j - y_j|^k$ is bounded, In particular $|x_j - y_j|$ goes to zero. From this one also derives that

$$\frac{|x_j|^k}{j^{3k}} + \frac{j|t_j - s_j|^2}{2} \leq C,$$

and then $|x_j| \leq C^{\frac{1}{k}} j^3$.

Moreover using Ishii's lemma [14], (see also lemma 2.1 in [3]) there exist $(X_j, Y_j) \in \mathcal{S}$ such that

$$\left(j(t_j - s_j),\ j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}},\ X_j + \frac{D^2(|x|^k)(x_j)}{j^{3k}}\right) \in \bar{J}^{2,+}u(x_j, t_j),$$

and

$$\big(j(t_j - s_j),\ j|x_j - y_j|^{k-2}(x_j - y_j), -Y_j\big) \in \bar{J}^{2,-}v(y_j, s_j).$$

Suppose that $x_j = y_j$. We prove then that $x_j \neq 0$. If it was the case the function $\varphi(x,t) = u(0, t_j) + \frac{j}{k}|x|^k + \frac{|x|^k}{j^{3k}} + \frac{j}{2}(t - s_j)^2 - \frac{j}{2}(s_j - t_j)^2$ would touch u by above on 0 and then using lemma 1 one would obtain since $k > \sup(2, \frac{\alpha+2}{\alpha+1})$

$$j(t_j - s_j) - 0 \leq g(0, t_j).$$

On the other hand since $v(0, s_j) - \frac{j}{2}(t - t_j)^2 + \frac{j}{2}(s_j - t_j)^2 - \frac{j|x|^k}{k}$ touches v by below on $(0, s_j)$, using once more lemma 1 we get

$$j(t_j - s_j) - 0 \geq f(0, s_j) + \frac{\kappa}{(T - s_j)^2}$$

Using $|t_j - s_j| \to 0$, the uniform continuity of f and g , substracting the two inequalities and passing to the limit we get a contradiction.

We now suppose that $x_j = y_j$ and we know that under this assumption, $x_j \neq 0$. Then the function $\varphi(x,t) = u(x_j, t_j) + \frac{j}{k}|x - x_j|^k + \frac{|x|^k}{j^{3k}} + \frac{j}{2}(t - s_j)^2 - \frac{j}{2}(s_j - t_j)^2 - \frac{|x_j|^k}{j^{3k}}$ achieves u by above on (x_j, t_j), where its gradient is different from 0. We then have

$$\begin{aligned} j(t_j - s_j) - F(x_j, \frac{k|x_j|^{k-2}x_j}{j^{3k}}, D^2(\frac{|x|^k}{j^{3k}})(x_j)) \quad &- \quad h(x_j, t_j) \cdot \frac{k|x_j|^{k-2}x_j}{j^{3k}} \left(\frac{|k|x_j|^{k-1}}{j^{3k}}\right)^\alpha \\ &\leq \quad g(x_j, t_j), \end{aligned}$$

On the other hand the function $\psi(x,t) = v(x_j, s_j) - \frac{j}{k}|x - x_j|^k - \frac{j}{2}(t - t_j)^2 + \frac{j}{2}(s_j - t_j)^2$ achieves v by below on (x_j, s_j).

Then one uses once more lemma 1, to get that

$$j(t_j - s_j) - 0 \geq f(x_j, s_j) + \frac{\kappa}{(T - s_j)^2},$$

We now use the properties of F to get that

$$\begin{aligned}
|F(x_j, \frac{k|x_j|^{k-2}x_j}{j^{3k}}, D^2\frac{(|x|^k)}{j^{3k}}(x_j))) &+ h(x_j,t_j)\cdot \frac{k|x_j|^{k-2}x_j}{j^{3k}}\left(\frac{k|x_j|^{k-1}}{j^{3k}}\right)^\alpha |\\
&\leq C\left(\frac{|x_j|^{k(\alpha+1)-\alpha-2}}{j^{3k(1+\alpha)}}+\frac{|x_j|^{(k-1)(\alpha+1)}}{j^{3k(1+\alpha)}}\right)\\
&\leq C\left(j^{3(k(\alpha+1)-\alpha-2)-3k(1+\alpha)}+j^{-3(\alpha+1)}\right)\\
&= o(1)
\end{aligned}$$

Finally using the fact that $|x_j - y_j| + |t_j - s_j|$ goes to zero, the uniform continuity of f and g , substracting the two equations and passing to the limit we get a contradiction.

We have obtained that $x_j \neq y_j$.

We now prove that $j^2|x_j - y_j|^{k-1} \to +\infty$. In particular this will imply that for j large enough $j|x_j-y_j|^{k-2}(x_j-y_j)+k\frac{|x_j|^{k-2}x_j}{j^{3k}} \neq 0$. Suppose by contradiction that for some constant $c > 0$, $j|x_j - y_j|^{k-1} \leq cj^{-1}$ then $|X_j| \leq j|x_j - y_j|^{k-2} \leq (j^2|x_j - y_j|^{k-1})^{\frac{k-2}{k-1}} j^{\frac{3-k}{k-1}} \to 0$ and also $|X_j| + |D^2\left(\frac{|x|^k}{j^{3k}}\right)(x_j)| \leq |X_j| + cj^{-6} \to 0$. Using the fact that u and v are respectively sub-and super-solution, one has

$$\begin{aligned}
g(x_j,t_j) &\geq j(t_j - s_j) - o(1)\\
\text{and } \frac{\kappa}{T^2} + f(y_j,s_j) &\leq j(t_j - s_j) + o(1).
\end{aligned}$$

Substracting the two inequalities, passing to the limit and using the properties of f and g, one gets a contradiction. We have obtained that $j|x_j - y_j|^{k-1} \geq \frac{c}{j}$ for some constant c. From this one derives that $j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}} \sim_{j\to+\infty} j|x_j - y_j|^{k-2}(x_j - y_j)$. With the aid of this remark and using the assumption $(H6)$

$$\begin{aligned}
| &\quad Fx_j,(j|x_j-y_j|^{q-2}(x_j-y_j)+\frac{k|x_j|^{k-2}x_j}{j^{3k}},X_j) - F(x_j, j|x_j-y_j|^{q-2}(x_j-y_j),X_j)|\\
\leq &\quad cj^{-3}|X_j|(j|x_j-y_j|^{k-1})^{\alpha-1}\\
\leq &\quad \begin{cases} cj^{-\alpha-1}|x_j-y_j|^{k-2} & \text{if } \alpha<1\\ cj^{-3+\frac{\alpha+1}{k}}(j|x_j-y_j|^k)^{\alpha-\frac{\alpha+1}{k}} & \text{if } \alpha\geq 1\end{cases}\\
= &\quad o(1)
\end{aligned}$$

by the choice of k. One also has using the assumption (H2)

$$\begin{aligned}
| \quad & F(x_j, j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}}, X_j + \frac{D^2(|x|^k)}{j^{3k}}(x_j)) \\
- \quad & F(x_j, j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}}, X_j)| \\
\leq \quad & cj^{-6}(j|x_j - y_j|^{k-1})^{\alpha} \\
\leq \quad & \begin{cases} cj^{-6-\alpha} & \text{if } \alpha < 0 \\ cj^{-6+\frac{\alpha}{k}}(j|x_j - y_j|^k)^{\alpha(\frac{k-1}{k})} & \text{if } \alpha \geq 0 \end{cases} \\
= \quad & o(1)
\end{aligned}$$

by the choice of k.

Treating analogously the terms involving h, in particular using the Hölder's regularity of h with respect to t, together with (H3), one obtains

$$\begin{aligned}
& g(x_j, t_j) \geq j(t_j - s_j) - F(x_j, j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}}, X_j + \frac{D^2(|x|^k)(x_j)}{j^{3k}}) \\
- \quad & h(x_j, t_j) \cdot \left(j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}}\right) \left|j|x_j - y_j|^{k-2}(x_j - y_j) + \frac{k|x_j|^{k-2}x_j}{j^{3k}}\right|^{\alpha} \\
\geq \quad & j(t_j - s_j) - F(x_j, j|x_j - y_j|^{k-2}(x_j - y_j), X_j) \\
- \quad & h(x_j, t_j) \cdot j|x_j - y_j|^{k-2}(x_j - y_j)|j|x_j - y_j|^{k-1}|^{\alpha} - o(1) \\
\geq \quad & j(t_j - s_j) - F(y_j, j|x_j - y_j|^{k-2}(x_j - y_j), -Y_j) \\
- \quad & h(y_j, s_j) \cdot j|x_j - y_j|^{k-2}(x_j - y_j)|j|x_j - y_j|^{k-1}(x_j - y_j)|^{\alpha} - o(1) \\
\geq \quad & f(y_j, s_j) + \frac{\kappa}{T^2} - o(1)
\end{aligned}$$

We now conclude as before : We use the fact that $|x_j - y_j| + |t_j - s_j|$ goes to zero, the uniform continuity of f and g , and we pass to the limit to get a contradiction.

This ends the proof of theorem 5.

We now prove that the solutions are Hölder's continuous when f and ψ are Hölder's continuous.

Theorem 6 *We assume that f is uniformly continuous and bounded, is γ_f Holderian with respect to t, uniformly in x, and that ψ is Holderian of exponent γ_ψ on $\mathbb{R}^N$ and bounded. Suppose that u is the solution of $1_{\{f,\psi\}}$ on $\mathbb{R}^N \times]0, T[$. Then u is Hölder's continuous of exponent γ_ψ with respect to x and of exponent $\gamma^\star = \inf(\gamma_f, \frac{1}{q(\alpha+1)-\alpha})$ with respect to t on every compact set of $\mathbb{R}^N \times]0, T[$.*

We shall need the following proposition, which proves some Hölder's regularity with respect to t, when x is fixed.

Proposition 10 *Under the assumptions of theorem 6 there exists some constant C_2 such that for all $x \in \mathbb{R}^N$ and for all $t, s > 0$*

$$|u(x, t + s) - u(x, t)| \leq C_2 s^{\gamma^\star}$$

where $\gamma^\star = \inf(\gamma_f, \frac{1}{q(\alpha+1)-\alpha})$, $q = \frac{q_1}{\gamma_\psi}$, $q_1 = \sup(2, \frac{\alpha+2}{\alpha+1})$.

Proof :

We first use the estimates (6.8) and (6.9) which give for $y = x$:

$$|\psi(x) - u(x, s)| \leq c_2 s^{\frac{1}{q(\alpha+1)-\alpha}}$$

and the comparison principle in Theorem 5 on $\mathbb{R}^N \times]0, T[$: We define for s fixed in $[0, T]$ and $t \in [0, T - s]$

$$v(x, t) = u(x, t + s) + c_f t s^{\gamma_f} + \sup_{x \in \mathbb{R}^N} |\psi(x) - u(x, s)|.$$

where c_f is some Hölder's constant of f with respect to t. Then v is a super-solution of $1_{\{f,\psi\}}$ on $\mathbb{R}^N \times [0, T-s[$. Let us note that v and u have the properties

$$u(x, t) \leq \psi(y) + c_1|x - y|^{\gamma_\psi} + c_2 t^{\frac{1}{q(\alpha+1)-\alpha}} \leq \psi(y) + 2c_1(|x - y| + 1) + c_2 T^{\frac{1}{q(\alpha+1)-\alpha}} \tag{6.12}$$

and

$$v(x, t) \geq \psi(y) - c_1|x - y|^{\gamma_\psi} - c_2(t+s)^{\frac{1}{q(\alpha+1)-\alpha}} \geq \psi(y) - 2c_1(|x - y| + 1) - c_2(2T)^{\frac{1}{q(\alpha+1)-\alpha}} \tag{6.13}$$

and $u(x, 0) \leq v(x, 0)$ by construction.

Hence one can apply the comparison theorem 5 to obtain that

$$u(x, t) \leq v(x, t) + \sup_{x \in \Omega} |u(x, 0) - v(x, 0)| \leq u(x, t + s) + c_f T s^{\gamma_f} + c_2 s^{\frac{1}{q(\alpha+1)-\alpha}}$$

In the same manner defining $v(x, t) = u(x, t+s) - c_f t s^{\gamma_f} - \sup_x |\psi(x) - u(x, s)|$ then u and v are super and sub-solution for the same equation, and then using theorem 5 one gets

$$u(x, t) \geq u(x, t + s) - c_f T s^{\gamma_f} - c_2 s^{\frac{1}{q(\alpha+1)-1}}.$$

The result follows.

Proof of theorem 6

First we observe that u is bounded, taking $y = x$ in the inequalities (6.10) and (6.11) and using the fact that ψ is bounded. We denote by L_ψ some constant such that for all $(x, y) \in \mathbb{R}^N$, $|\psi(x) - \psi(y)| \le L_\psi |x - y|^{\gamma_\psi}$.

Let δ be given less than 1, $L > \sup(4c_1 + L_\psi, \left(\frac{4|f|_\infty}{\gamma_\psi^{1+\alpha}(1-\gamma_\psi)},\right)^{\frac{1}{1+\alpha}}, 1)$ and $M \ge \sup(\frac{2\sup u}{\delta^{\gamma^\star}}, c_2, \frac{2c_2 T^{\gamma^\star}}{\delta^{\gamma^\star}})$. We define the set

$$\Delta_\delta = \{(x, y, t, s), |x - y| < \delta, |t - s| < \delta, (t, s) \in]0, T[\}$$

and for j large (in particular $j \ge \frac{8\sup u}{\gamma_\psi}$), the function

$$\psi_j(x, y, t, s) = u(x, t) - u(y, s) - L|x - y|^{\gamma_\psi} - \frac{|x|^2}{2j^2} - M|t - s|^{\gamma^\star}.$$

We shall prove that for j large enough, ψ_j is ≤ 0. The result will follow by passing to the limit on each compact set of $\mathbb{R}^N \times]0, T[$.

We then assume by contradiction that ψ_j has a maximum strictly positive. Then for κ small enough

$$\psi_j - \frac{\kappa}{T - t} - \frac{\kappa}{T - s}$$

has also its supremum strictly positive and we begin to observe that on the boundary of Δ_δ, this function is ≤ 0.

Indeed in the case where $|t - s| = \delta$ then by hypothesis (6.10) and (6.11)

$$u(x, t) - u(y, s) \le c_1|x - y|^{\gamma_\psi} + 2c_2 T^{\gamma^\star} \le L|x - y|^{\gamma_\psi} + M|t - s|^{\gamma^\star}$$

In the case where $t = 0$, $s > 0$ and $|x - y| \le \delta$ one uses once more (6.10) and (6.11). Finally the supremum cannot be achieved for $t = T$ or $s = T$.
Let us note that if ψ_j has a supremum > 0,

$$\psi_j^n(x, t, y, s) = u(x, t) - u(y, s) - L|x - y|^{\gamma_\psi} - \frac{|x|^2}{2j^2} - M(\frac{1}{n^2} + [t - s|^2)^{\frac{\gamma^\star}{2}} - \frac{\kappa}{T - t} - \frac{\kappa}{T - s}$$

has also a supremum > 0 achieved inside Δ_δ, for n large enough. We fix n large enough. Let (x_j, y_j, t_j, s_j) be a point where the supremum of ψ_n is achieved.

By the previous considerations, it cannot be achieved on the boundary. By proposition 10 one has $x_j \neq y_j$ and then the function $(x,y) \mapsto |x-y|^{\gamma_\psi}$ is $\mathcal{C}^2$ on a neighborhood of (x_j, y_j). Using Ishii's lemma (see also Lemma 2.1 in [3]) we have the existence of (X_j, Y_j) with

$$\Big(\gamma^\star M(t_j-s_j)(\frac{1}{n^2}+|t_j-s_j|^2)^{\frac{\gamma^\star}{2}-1}+\frac{\kappa}{(T-t_j)^2},$$
$$\gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2}+\frac{x_j}{j^2}, X_j+\frac{I}{j^2}\Big) \in \bar{J}^{2,+}u(x_j,t_j)$$

$$\begin{aligned}(\gamma^\star M(t_j-s_j)(\frac{1}{n^2}+|t_j-s_j|^2)^{\frac{\gamma^\star}{2}-1} &- \frac{\kappa}{(T-s_j)^2}, \gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2}, -Y_j) \\ &\in \bar{J}^{2,-}u(y_j,s_j)\end{aligned}$$

with
$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \le \begin{pmatrix} B(x_j,y_j) & -B(x_j,y_j) \\ -B(x_j,y_j) & B(x_j,y_j) \end{pmatrix}$$
with $B(x,y) = L\gamma_\psi |x-y|^{\gamma_\psi-2}(I+(\gamma_\psi-2)\frac{(x-y)\otimes(x-y)}{|x-y|^2})$

Let us observe that due to the hypothesis, $|\frac{x_j}{j^2}| \le \frac{2\sup u}{j} \le \frac{\gamma_\psi L\delta^{\gamma_\psi-1}}{2}$, and then $|\gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2}+\frac{x_j}{j^2}| \ge \frac{\gamma_\psi}{2}L|x_j-y_j|^{\gamma_\psi-1}$.

We use as in the proof of theorem 2, the inequality

$$|tr(X_j+Y_j)| = -tr(X_j+Y_j) \ge 4\gamma_\psi(1-\gamma_\psi)L|x_j-y_j|^{\gamma_\psi-2}$$

and the fact that for some universal constant c

$$|X_j|+|Y_j| \le c(|tr(X_j+Y_j)|$$

We then use the property (H6) of F to get that

$$\begin{aligned}|F(x_j,\gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2} \quad & +\frac{x_j}{j^2}, X_j+\frac{I}{j^2}) \\ &- F(x_j,\gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2}, X_j)| \\ &\le O(j^{-1})(L|x_j-y_j|^{\gamma_\psi-1})^{\alpha-1}|X_j| \\ &+ O(\frac{1}{j^2})(L|x_j-y_j|^{\gamma_\psi-1})^{\alpha} \\ &\le o(1)(L|x_j-y_j|^{\gamma_\psi-1})^\alpha|tr(X_j+Y_j)|.\end{aligned}$$

And we use only the fact that h is bounded to observe that

$$\begin{aligned}|h(x_j,t_j)-h(y_j,s_j) &\cdot (\gamma_\psi L)^{1+\alpha}(x_j-y_j)|x_j-y_j|^{(1+\alpha)(\gamma_\psi-1)-1}| \\ &\le o(1)(\gamma_\psi L|x_j-y_j|^{\gamma_\psi-1})^\alpha |tr(X_j+Y_j)|\end{aligned}$$

We now write

$$\begin{aligned}f(x_j,t_j) &\ge \gamma^\star M(t_j-s_j)(\frac{1}{n^2}+|t_j-s_j|^2)^{\frac{\gamma^\star}{2}-1}+\frac{\kappa}{(T-t_j)^2} \\ &- F(x_j,\gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2}+\frac{x_j}{j^2},X_j+\frac{I}{j^2}) \\ &- h(x_j,t_j)\cdot(\gamma_\psi L)^{1+\alpha}(x_j-y_j)|x_j-y_j|^{(1+\alpha)(\gamma_\psi-1)-1} \\ &\ge \gamma^\star M(t_j-s_j)(\frac{1}{n^2}+|t_j-s_j|^2)^{\frac{\gamma^\star}{2}-1}-\frac{\kappa}{(T-s_j)^2} \\ &- F(y_j,\gamma_\psi L(x_j-y_j)|x_j-y_j|^{\gamma_\psi-2},-Y_j) \\ &- h(y_j,s_j)\cdot(\gamma_\psi L)^{1+\alpha}(x_j-y_j)|x_j-y_j|^{(1+\alpha)(\gamma_\psi-1)-1} \\ &\quad +(\gamma_\psi L|x_j-y_j|^{\gamma_\psi-1})^\alpha tr(X_j+Y_j)+o(1)|\gamma_\psi L|x_j-y_j|^{\gamma_\psi-1}|^\alpha(|tr(X_j+Y_j)|) \\ &\ge f(y_j,s_j)+(\gamma_\psi L|x_j-y_j|^{\gamma_\psi-1})^\alpha tr(X_j+Y_j)(1-o(1)).\end{aligned}$$

We have obtained a contradiction since this would imply that

$$(\gamma_\psi L|x_j-y_j|^{\gamma_\psi-1})^\alpha L|x_j-y_j|^{\gamma_\psi-2}(1-o(1))\le 2|f|_\infty,$$

which is absurd by the choice of the constant L.

With the same arguments, and using Ascoli's theorem, we have also

Corollary 2 *Let $(f_n,\, \psi_n)$ be a sequence of bounded Hölder's continuous functions, ψ_n being uniformly Hölder's and (f_n) being uniformly Hölder's in t, uniformly w.r.t. x. Then the sequence (u_n) of solutions of $1_{\{f_n,\psi_n\}}$ is relatively compact in $\mathcal{C}_c(\mathbb{R}^N\times]0,T[)$.*

7 Appendix

In this appendix we prove that the solutions of Ohnuma and Sato in the case where $\alpha\neq 0$ are the same as our solutions. In the same manner we prove that it is also the case for the infinity Laplacian using the adapted definition of Juutinen and Kawhol.

7.1 The equivalence with the definition of Ohnuma and Sato in the case of the p-Laplacian.

For the convenience of the reader we recall here the definition of $\mathcal{F}(F)$ and $\mathcal{A}(F)$, [20]. Here h and the right hand side are assumed to be zero, and the operator F does not depends on x.

Definition 2 *We denote by $\mathcal{F}(F)$ the set of functions $\varphi \in \mathcal{C}^2([0,\infty[)$ which satisfies*

$$\varphi(0) = \varphi'(0) = \varphi^{"}(0) = 0,\ \varphi^{"}(r) > 0 \text{ for all } r > 0$$

and

$$\lim_{|x|\to 0, x\neq 0} F(\pm\nabla\varphi(|x|), \pm D^2\varphi(|x|)) = 0$$

A function $\psi \in \mathcal{C}^2(Q_T) \in \mathcal{A}(F)$ if for all $\bar{z} = (\bar{x}, \bar{t}) \in Q_T$ with $\nabla\psi(\bar{z}) = 0$ there are a constant $\delta > 0$, some $\varphi \in \mathcal{F}(F)$, and $\omega \in \mathcal{C}([0,\infty[)$ satisfying $\omega \geq 0$ and $\lim_{r\to 0} \frac{\omega(r)}{r} = 0$ and thar for all $z = (t, x)$, $|z - \bar{z}| < \delta$, satisfies

$$|\psi(z) - \psi(\bar{z}) - \psi_t(\bar{z})(t - \bar{t})| \leq \varphi(|x - \bar{x}) + \omega(|t - \bar{t}|)$$

And we recall the definition of viscosity solution

Definition 3 *u is a viscosity sub-solution if for all $\psi \in \mathcal{A}(F)$ such that ψ achieves u by above on $\bar{z} = (\bar{x}, \bar{t})$*

$$\begin{cases} \psi_t(\bar{z}) - F(\nabla\psi, D^2\psi)(\bar{z}) \leq 0, & \text{if } \nabla\psi(\bar{z}) \neq 0 \\ \psi_t(\bar{z}) \leq 0 & \text{otherwise} \end{cases}$$

Proposition 11 *The solutions in the sense of definition 1 are the same as the solutions in the Ohnuma and Sato's sense.*

Proof

Suppose that u is a super-solution of $1_{\{0\}}$ in the Ohnuma and Sato's sense. Suppose that $(\bar{x}, \bar{t})$ is some point such that for some δ_1 and for some $\mathcal{C}^1$ function φ on $]0, T[$:

$$\inf_{|t-\bar{t}|<\delta_1} (u(\bar{x}, t) - \varphi(t)) = u(\bar{x}, \bar{t}) - \varphi(\bar{t}) = 0$$

and such that $x \mapsto \inf_{|t-\bar{t}|<\delta_1}(u(x,t)-\varphi(t))$ is constant on $B(\bar{x},\delta)$ for some $\delta > 0$. Then in particular

$$\inf_{x\in B(\bar{x},\delta),|t-\bar{t}|<\delta_1}(u(x,t)-\varphi(t))$$

has its infimum equals to zero achieved on $(\bar{x},\bar{t})$. Then, for $\epsilon > 0$ the function

$$h(x,t)=\varphi(\bar{t})+\varphi'(\bar{t})(t-\bar{t})+\frac{1}{2}(\varphi^{"}(\bar{t})-\epsilon)(t-\bar{t})^2$$

which belongs to $A(f)$, [20], satisfies

$$\inf_{(|t-\bar{t}|<\delta_1,x\in B(\bar{x},\delta)}(u(x,t)-h(x,t))=0$$

Indeed

$$\inf_{|t-\bar{t}|<\delta_1,x\in B(\bar{x},\delta)}(u-h)(x,t)\leq u(\bar{x},\bar{t})-\varphi(\bar{t})=0.$$

Moreover for t close to $\bar{t}$

$$\varphi(t)\geq\varphi(\bar{t})+\varphi'(\bar{t})(t-\bar{t})+\frac{1}{2}(\varphi^{"}(\bar{t})-\epsilon)(t-\bar{t})^2$$

hence

$$\inf_{(|t-\bar{t}|<\delta_1,x\in B(\bar{x},\delta)}(u-h)(x,t)\geq\inf_{|t-\bar{t}|<\delta_1,x\in B(\bar{x},\delta)}(u(x,t)-\varphi(t))$$

and then since u is a super-solution of $1_{\{0\}}$, $\varphi'(\bar{t})\geq 0$ which is the desired conclusion.

We want to prove the reverse sense. We assume that u is a super solution in our sense. We suppose that $(\bar{x},\bar{t})$ and ψ are such that $(u-\psi)\geq(u-\psi)(\bar{x},\bar{t})=0$, with $\psi\in\mathcal{A}(F)$.

Let $\varphi\in\mathcal{F}(F)$ and ω be a continuous function such that $\omega(0)=0$, $\omega(t-\bar{t})=o(|t-\bar{t}|)$, be such that for $(x,t)\in V$ a neighborhood of $(\bar{x},\bar{t})$,

$$|\psi(x,t)-\psi(\bar{x},\bar{t})-\partial_t\psi(\bar{x},\bar{t})(t-\bar{t})|\leq\varphi(|x-\bar{x}|)+\omega(t-\bar{t})$$

Then

$$h(x,t):=\psi(\bar{x},\bar{t})+\partial_t\psi(\bar{x},\bar{t})(t-\bar{t})-\varphi(|x-\bar{x}|)-\omega(t-\bar{t})\leq\psi(x,t)$$

Moreover

$$\inf_{(x,t)\in V}(u(t,x)-h(x,t))=0$$

Indeed

$$\inf_{(x,t)\in V}(u(x,t)-h(x,t))\leq u(\bar{x},\bar{t})-h(\bar{x},\bar{t})$$

secondly by the previous remark,

$$u-h\geq u-\psi.$$

Now acting as in lemma 1 ie replacing $C_1|x-\bar{x}|^k$ by $\varphi(|x-\bar{x}|)$ and $C_2|t-\bar{t}|^2$ by $\omega(|t-\bar{t}|)$ one gets since $\lim_{x\to 0}F(\nabla f,D^2f)(|x|)=0$ that $\partial_t\varphi(\bar{x},\bar{t})\geq 0$, which is the desired conclusion.

7.2 The case of the infinity Laplacian

We prove here that our definition is equivalent to the one of Juutinen and Kawhol in the case of the infinity Laplacian ([18]). Here $f=0$, $h=0$ and the operator F is replaced by F_∞ defined as $F_\infty(M,p)=\langle M\frac{p}{|p|},\frac{p}{|p|}\rangle$. We denote by $\lambda(M)$ and $\Lambda(M)$ the smallest and the larger eigenvalue of M. The definition of viscosity solution is the following

Definition 4 *u is a viscosity super-solution of*

$$u_t-F_\infty(D^2u,Du)=0$$

if for all $(\bar{x},\bar{t})\in Q_T$ and for all $\varphi\in\mathcal{C}^{1,2}$ with $(u-\varphi)\geq(u-\varphi)(\bar{x},\bar{t})$ on a neighborhood of $(\bar{x},\bar{t})$ and $\nabla\varphi(\bar{x},\bar{t})\neq 0$, then

$$\varphi_t(\bar{x},\bar{t})-F_\infty(D^2\varphi,D\varphi)(\bar{x},\bar{t})\geq 0$$

and if $\nabla\varphi(\bar{x},\bar{t})=0$

$$\varphi_t(\bar{x},\bar{t})-\lambda(D^2\varphi)(\bar{x},\bar{t})\geq 0.$$

u is a viscosity sub-solution of

$$u_t-F_\infty(D^2u,Du)=0$$

if for all $(\bar{x},\bar{t})\in Q_T$ and for all $\varphi\in\mathcal{C}^{1,2}$ with $(u-\varphi)\geq(u-\varphi)(\bar{x},\bar{t})$ around $(\bar{x},\bar{t})$and $\nabla\varphi(\bar{x},\bar{t})\neq 0$, then

$$\varphi_t(\bar{x},\bar{t})-F_\infty(D^2\varphi,D\varphi)(\bar{x},\bar{t})\leq 0$$

and if $\nabla\varphi(\bar{x},\bar{t}) = 0$

$$\varphi_t(\bar{x},\bar{t}) - \Lambda(D^2\varphi)(\bar{x},\bar{t}) \leq 0.$$

We treat only the equivalence between the super-solutions in our sense and in the sense of [18]. For this, we shall need the following lemma, whose proof is postponed for the sake of clearness.

Lemma 2 *Suppose that* u *is a super-solution in our sense of*

$$u_t - F_\infty(x, \nabla u, D^2 u) \geq 0$$

and suppose that φ *is some* $\mathcal{C}^1$ *function on* $]0,T[$, *with* $\varphi(\bar{t}) = 0$, *that* $k > \sup(2, \frac{\alpha+2}{\alpha+1})$, *that* M *is some symmetric matrix and* $(0,\bar{t}) \in \Omega\times]0,T[$ *are such that for some* $\delta_1 > 0$

$$\inf_{x\in B(0,\delta_1), |t-\bar{t}|<\delta_1} (u(x,t) - \varphi(t) - \frac{1}{2}(Mx,x)) = u(0,\bar{t})$$

Then

$$\varphi'(\bar{t}) - \lambda(M) \geq 0.$$

We postpone the proof of Lemma 2.

We prove first that if u is a super-solution in the sense of [18], it is a super-solution in our sense. If φ is some function which achieves u by below on $(\bar{x},\bar{t})$ and is such that $\nabla\varphi(\bar{x},\bar{t}) \neq 0$, there is nothing to prove. Suppose now that there exists δ_1 and some $\mathcal{C}^1$ function on $[\bar{t}-\delta_1, \bar{t}+\delta_1[$ such that for $x \in B(\bar{x},\delta_1)$

$$\inf_{|t-\bar{t}|\leq\delta_1} (u(x,t) - \varphi(t)) = u(\bar{x},\bar{t}) - \varphi(\bar{t})$$

then $(t,x) \mapsto \varphi(t)$ achieves u by below and then the Hessian of φ is zero, hence

$$\varphi'(\bar{t}) - 0 \geq 0$$

which is the same as our conclusion.

We prove the reverse assertion.

We then consider a super-solution u in our sense and assume that φ is some $\mathcal{C}^2$ function which achieves u by below on $(\bar{x},\bar{t})$ with $\nabla\varphi(\bar{x},\bar{t}) = 0$. We apply lemma 2 with $\bar{x}$ in place of 0, $\nabla\varphi(\bar{x},\bar{t})=0$ and replacing $\varphi(t)$by $\partial_t\varphi(\bar{x},\bar{t})(t-\bar{t})$, and $M=D^2\varphi(\bar{x},\bar{t})$ one gets the desired conclusion.

Proof of lemma 2:

For $C_2>0$ one still has

$$\inf_{x\in B(0,\delta_1),|t-\bar{t}|<\delta_1}(u(x,t)-\varphi(t)-\frac{1}{2}(Mx,x)+C_2(t-\bar{t})^2)=u(0,\bar{t})$$

and the infimum is strict in t.

We assume first that $x\mapsto\inf_{|t-\bar{t}|<\delta_1}(u(x,t)-\varphi(t)+C_2(t-\bar{t})^2)$ is equal to $u(0,\bar{t})$ and is constant w.r.t. x in a neighboorhood of $\bar{x}$. We then prove that $M\le 0$ and $\varphi'(\bar{t})\ge 0$, which will imply that $\varphi'(\bar{t})-\lambda(M)\ge 0$.

Indeed one has for all x in a neighborhood of 0, $u(0,\bar{t})=\inf_{|t-\bar{t}|<\delta_1}(u(x,t)-\varphi(t)+C_2(t-\bar{t})^2)$ and also by hypothesis

$$u(0,\bar{t})=\inf_{(|t-\bar{t}|<\delta_1),x\in B(0,\delta_1)}\{u(x,t)-\varphi(t)-\frac{1}{2}\langle Mx,x\rangle+C_2(t-\bar{t})^2\}$$

and then for all x in a neighborhood of 0,

$$u(0,\bar{t})\le\inf_{|t-\bar{t}|<\delta_1}\{u(x,t)-\varphi(t)+C_2(t-\bar{t})^2\}-\frac{1}{2}\langle Mx,x\rangle=u(0,\bar{t})-\frac{1}{2}\langle Mx,x\rangle$$

This implies that for all x in a neighborhhod of 0,

$$\langle Mx,x\rangle\le 0,$$

or equivalently that M is a nonpositive symmetric matrix. Using the definition, as we pointed out before, $\varphi'(\bar{t})\ge f(0,\bar{t})$ and this implies the desired result.

We now assume that we are not in the case where $x\mapsto\inf_{|t-\bar{t}|<\delta_1)}(u(x,t)-\varphi(t)+C_2(t-\bar{t})^2)$ is equal to $u(0,\bar{t})$ and is constant w.r.t. x in a neighboorhood of $\bar{x}$.

For the sequel one can assume that M is invertible. indeed, if it is not the case there exists $\epsilon>0$ small in order that $M-\epsilon Id$ is invertible. Moreover $M-\epsilon Id$ is also such that

$$\inf_{(|t-\bar{t}|<\delta_1),x\in B(0,\delta_1)}\{u(x,t)-\varphi(t)-\frac{1}{2}\langle(M-\epsilon Id)(x),x\rangle+C_1|x|^k+C_2(t-\bar{t})^2\}=u(0,\bar{t})$$

So we shall prove that

$$\varphi'(\bar{t})-\lambda(M-\epsilon Id)\ge 0$$

and we shall get the result by passing to the limit with ϵ.

So from now we assume that M is invertible.

For $k>2$ and for all positive constant C_1 then

$$\inf_{|t-\bar{t}|<\delta_1, x\in B(0,\delta_1)} \{u(x,t)-\varphi(t)-\frac{1}{2}\langle Mx,x\rangle + C_1|x|^k + C_2(t-\bar{t})^2\}$$

has also its infimum achieved on $(0,\bar{t})$, and this infimum is strict in x and t. Hence for all $\delta>0$ there exists $\epsilon(\delta)>0$ such that

$$\begin{aligned}\inf\Bigg(\inf_{|t-\bar{t}|>\delta, x\in B(0,\delta_1)} & \{u(x,t)-\varphi(t)-\frac{1}{2}\langle Mx,x\rangle + C_1|x|^k + C_2(t-\bar{t})^2\} \\ & \inf_{(|t-\bar{t}|<\delta_1, |x|>\delta} \{u(x,t)-\varphi(t)-\frac{1}{2}\langle Mx,x\rangle + C_1|x|^k + C_2(t-\bar{t})^2\}\Bigg) \\ > & \ u(0,\bar{t})+\epsilon(\delta)\end{aligned}$$

In the following we choose δ such that $(2\delta)^{k-1} < \frac{\inf_{\lambda_i\in Sp(M)}|\lambda_i(M)|}{2kC_1}$. Let then δ_2 be such that $\delta_2<\delta$ and

$$k(2\delta_1)^{k-1}C_1\delta_2 + |M|_\infty(\delta_2^2+2\delta_2\delta_1) \le \epsilon/4$$

With this choice, using the fundamental calculus theorem, one gets that for $x\in B(0,\delta_2)$,

$$\begin{aligned}\inf_{\{|t-\bar{t}|<\delta), y\in B(0,\delta)\}} & \{u(y,t)-\varphi(t)-\frac{1}{2}\langle M(y-x),(y-x)\rangle + C_1|x-y|^k + C_2(t-\bar{t})^2\} \\ \le & \inf_{(|t-\bar{t}|<\delta_1), y\in B(0,\delta_1)} \{u(y,t)-\varphi(t)-\frac{1}{2}\langle My,y\rangle + C_1|y|^k + C_2(t-\bar{t})^2\} + \frac{\epsilon}{4} \\ = & \ u(0,\bar{t})+\frac{\epsilon}{4} \qquad (7.14)\end{aligned}$$

while

$$\begin{aligned}\inf\Bigg(\inf_{\{|t-\bar{t}|<\delta_1), |y|>\delta\}} & (\{u(y,t)-\varphi(t)-\frac{1}{2}\langle M(y-x),(y-x)\rangle + C_1|x-y|^k + C_2(t-\bar{t})^2\}, \\ & \inf_{(|t-\bar{t}|>\delta, y\in B(0,\delta_1)} \{u(y,t)-\varphi(t)-\frac{1}{2}\langle M(y-x),(y-x)\rangle \\ + & \ C_1|x-y|^k + C_2(t-\bar{t})^2\}\Big) \\ \ge & \ u(0,\bar{t})+\frac{3\epsilon}{4} \qquad (7.15)\end{aligned}$$

We choose x_δ as follows : Since the function $\inf_{|t-\bar{t}|<\delta_1)}(u(x,t)-\varphi(t)+C_2|t-\bar{t}|^2)$ is not constant around $\bar{x}$, for all $\delta>0$ there exists x_δ and y_δ in $B(0,\delta_2)$ such that

$$\begin{aligned}\inf_{|t-\bar{t}|<\delta_1}\{u(x_\delta,t)-\varphi(t) &+ C_2|t-\bar{t}|^2\}\\ &> \inf_{|t-\bar{t}|<\delta_1}\{u(y_\delta,t)-\varphi(t)+C_2|t-\bar{t}|^2\\ &- \left(\frac{1}{2}\langle M(x_\delta-y_\delta),x_\delta-y_\delta\rangle\right)+C_1|x_\delta-y_\delta|^k\}.\end{aligned}$$

Then the infimum $\inf_{(|t-\bar{t}|<\delta_1),y\in B(0,\delta_1)}(u(y,t)-\varphi(t)-\frac{1}{2}\rangle M(y-x_\delta),(y-x_\delta)\rangle+C_1|x_\delta-y|^k+C_2(t-\bar{t})^2)$, is achieved on some point (z_δ,t_δ) with $z_\delta\neq x_\delta$. Moreover by (7.14) and (7.15) the infimum is achieved in $B(0,\delta)\times]\bar{t}-\delta,\bar{t}+\delta[$. Let (z_δ,t_δ) be a point on which this infimum is achieved, then

$$\begin{aligned}\psi(x,t) &= \varphi(t)+\frac{1}{2}\langle M(x-x_\delta),x-x_\delta)\rangle-\frac{1}{2}\langle M(z_\delta-x_\delta),(z_\delta-x_\delta)\rangle\\ &+ C_1|x_\delta-z_\delta|^k-C_1|x_\delta-x|^k\\ &- C_2(t-\bar{t})^2+C_2(t_\delta-\bar{t})^2\end{aligned}$$

achieves u by below on (z_δ,t_δ).

With the choice of δ, the gradient of ψ on z_δ, which equals $M(z_\delta-x_\delta)+kC_1|x_\delta-z_\delta|^{k-2}(x_\delta-z_\delta)$ is different from zero, since $z_\delta\neq x_\delta$. Indeed if it was the case, $x_\delta-z_\delta$ would be an eigenvector for M corresponding to the eigenvalue $kC_1|x_\delta-z_\delta|^{k-1}$, which is impossible since $kC_1(2\delta)^{k-1}<\inf_i(|\lambda_i(M)|$. Using the fact that u is a super-solution one gets that

$$\begin{aligned}\varphi'(t_\delta) &- F_\infty(M(z_\delta-x_\delta)+kC_1|x_\delta-z_\delta|^{k-2}(x_\delta-z_\delta),M-C_1D^2(|x_\delta-z|^k)(z_\delta))\\ &\geq 0\end{aligned}$$

and then using the definition of F_∞ one has $F_\infty(M(z_\delta-x_\delta)+kC_1|x_\delta-z_\delta|^{k-2}(x_\delta-z_\delta),M-C_1D^2(|x_\delta-z|^k)(z_\delta))\geq\lambda(M)+o(1)$ and finally

$$\begin{aligned}\varphi'(t_\delta) &- \lambda(M)\\ &\geq 0.\end{aligned}$$

Letting δ go to zero and using $z_\delta\in B(0,\delta_2)\subset B(0,\delta)$, $|t-t_\delta|<\delta$, $k>2$ and the lower semicontinuity of f one gets

$$\varphi'(\bar{t})-\lambda(M)\geq 0.$$

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