Compactness Theorems for Spaces of Functions with Bounded Derivatives and Applications to Limit Analysis Problems in Plasticity

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Introduction

This article presents certain results on variational problems that arise in different areas of mechanics and, especially, in plasticity. The common mathematical feature of these problems is that they involve energies that are coercive on non-reflexive spaces of the type L^1, so the solutions, when they exist, lie in generalized spaces such as

$$\mathrm{BV}(\Omega) = \{u \in L^1(\Omega), \nabla u \in M^1(\Omega, \mathbb{R}^N)\},$$

$$\mathrm{BD}(\Omega) = \{u \in L^1(\Omega), \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \in M^1(\Omega, E)\},$$

or

$$\mathrm{HB}(\Omega) = \{u \in L^1(\Omega), u_{ij} = (\nabla\nabla u)_{ij} \in M^1(\Omega, R)\},$$

where Ω is a bounded open set in $\mathbb{R}^N$, E is the space of symmetric tensors of order two on $\mathbb{R}^N$, and $M^1(\Omega, X)$ denotes the space of bounded measures on Ω with values in Y. The use of these spaces settles in a satisfactory way the problem of existence of solutions in the sense of the calculus of variations. However, it leaves open questions that are important in the applications to mechanics. To fix ideas, let us consider the problem of limit analysis for antiplane shear in plasticity, namely

$$Lu = \inf_{\substack{u \in \mathrm{BV}(\Omega) \\ \int_\Omega fu + \int_{\Gamma_1} Fu = 1}} \left\{ \int_\Omega |\nabla u| + \int_{\Gamma_0} |u\vec{n}| \right\}$$

where $f \in L^N(\Omega)$ is the body force and $F \in L^\infty(\Gamma_1)$ is the traction on the boundary. The methods of the calculus of variations guarantee that a minimizing sequence contains a subsequence (u_n) which converges, weakly in $\mathrm{BV}(\Omega)$, to a minimizer $u \in \mathrm{BV}(\Omega)$. However, this does not allow us to conclude that u satisfies the work condition $Lu = 1$ since we do not know, for instance, that $u_{n|\Gamma} \to u|_\Gamma$ in any sense.

Questions of this type have motivated the first part of this article which seeks to establish sufficient conditions for compactness of the known injections $\mathrm{BV}(\Omega) \hookrightarrow L^{N/N-1}(\Omega)$, $\mathrm{BD} \hookrightarrow L^{N/N-1}(\Omega)$, $\gamma_0(\mathrm{BV}) \hookrightarrow L^1(\Gamma)$, $\gamma_0(\mathrm{BD}) \hookrightarrow L^1(\Gamma)$, $\mathrm{HB} \hookrightarrow C(\Omega)$.

These injections, which are of course continuous in the strong topologies may be continuous in other, intermediate, topologies which are induced by convex functions of a measure defined by R. TÉMAM and the author in [8]. To be precise, when ψ is a convex lower semi-continuous proper and non-negative function which is linear at infinity, we may define on $X_s = \{u \in L^1(\Omega), Su \in M^1(\Omega)\}$ the distance

$$d_{\psi,s}(u, v) = |u - v|_{L^1} + \left| \int \psi(Su) - \psi(Sv) \right|$$

where S is one of the operators ∇, ε or $\nabla\nabla$ and $\psi(Su)$ is the bounded measure defined in [8]. The central result in Part I is that if (u_n) is a sequence converging to u in the metric $d_{\psi,s}$ then, if $S = \nabla$ or ε, u_n tends to u in $L^{N/N-1}(\Omega)$, and $u_{n|\Gamma}$ tends to $u_{|\Gamma}$ in $L^1(\Gamma)$; if $S = \nabla\nabla$, u_n is pointwise convergent to u.

These results are of general mathematical interest, independent of the intended applications, and so Part I of the paper may be read independently of the rest.

The second part of the paper deals with applications of the above results. In the first place we obtain a solution of the problem of relaxed limit analysis in plasticity, under weak assumptions. Furthermore, we provide improvements to the existence theory in plasticity, by KOHN & TÉMAM [12]. In particular, we show that the minimizing sequence (u_n) for the energy actually converges in L^{N^*}, thus answering a question raised by STRANG & TÉMAM [21]. Moreover, we show that the trace of (u_n) on the part Γ_1 of the boundary on which traction is prescribed tends, in L^1, to the trace of the limit. This solves a problem posed by SUQUET in his thesis [19] and it also provides a possible explanation to the observation that the displacement gradient does not jump across the interface formed by the clamped and the free part of the boundary of the body.

Part I. Compactness in BV, BD and HB Spaces

As I mentioned in the general introduction, my aim in this chapter is to describe a family of convenient topologies on the spaces BV, BD, HB, which make the following injections BD, BV $\hookrightarrow L^{N/N-1}(\Omega)$, $\gamma_0(\mathrm{BV}, \mathrm{BD}) \hookrightarrow L^1(\Gamma)$, HB $\hookrightarrow \mathscr{C}(\bar{\Omega})$ continuous.

I.1. Survey of known properties and notations

Let Y be a real Euclidean finite dimensional space, Ω be an open bounded set of $\mathbb{R}^N$, $N \geqq 2$, with boundary Γ. Suppose that ψ is a convex lower semi-continuous proper function from Y into $\overline{\mathbb{R}}$ such that

$$\psi(0) = 0, \psi \geqq 0 \tag{1.1}$$

which is linear at infinity, *i.e.* there are positive constants c_0, c_1 with

$$(c_0 |\xi| - 1) \leqq \psi(\xi) \leqq c_1(|\xi| + 1) \tag{1.2}$$

for all ξ in Y. It is shown in [8] that (1.2) is equivalent to

$$K = \operatorname{dom} \psi^* \text{ is bounded and contains 0 in its interior} \tag{1.3}$$

where ψ^* denotes the conjugate of ψ,

$$\psi^*(\xi) = \sup_{\eta \in Y} \{\xi \cdot \eta - \psi(\eta)\}$$

(see for instance J. J. Moreau [15], R. T. Rockafellar [17] or I. Ekeland & R. Témam [9]). We define the asymptotic function

$$\psi_\infty(\xi) = \lim_{t \to +\infty} \frac{\psi(t\xi)}{t} \tag{1.4}$$

which coincides with the conjugate of the indicator function of K:

$$\psi_\infty(\xi) = \sup_{\eta \in K} \xi : \eta\,. \tag{1.5}$$

It is shown in [8] that for μ in the space $M^1(\Omega, Y)$ of bounded measures with values in Y we may define a bounded measure $\psi(\mu)$ which coincides with $\psi \circ g$ when $\mu = g\,dx$ in an ordinary sense. We recall a proposition established in [8].

Lemma 1.1. *Let (μ_j) be a sequence in $M^1(\Omega, Y)$ which converges weakly (vaguely) to a measure μ and let ψ be a convex function as above; then for a subsequence still denoted (μ_j), $\psi(\mu_j)$ converges weakly to a measure ν in $M^1(\Omega)$ and*

$$\psi(\mu) \leqq \nu\,. \tag{1.9}$$

Another interesting fact is the following approximation (*cf.* [8]).

Lemma 1.2. *For every μ in $M^1(\Omega, Y)$, there is a sequence of functions (u_j) in $\mathscr{C}_0^\infty(\Omega, Y)$ such that for every convex function ψ satisfying* (1.1) *and* (1.2):

$$\psi(u_j) \to \psi(\mu) \quad \textit{tightly on } \Omega,$$

i.e.

$$\int \psi(u_j)\,\varphi \to \int \psi(\mu)\,\varphi \quad \textit{for every } \varphi \textit{ in } \mathscr{C}(\bar{\Omega}).$$

Remark 1.1. This result easily enables us to extend the inequality (1.2) to bounded measures on Ω. In other words, for each μ in $M^1(\Omega, Y)$, we have

$$c_0(|\mu| - 1) \leqq \psi(\mu) \leqq c_1(|\mu| + 1)\,. \tag{1.10}$$

We now recall two lemmas on *Concentration Compactness* (*cf.* P.-L. LIONS [13]).

For $d \geqq 1$, we define the space

$$\nabla^d B(\Omega) = \{u \in L^1(\Omega), \nabla^d u \in M^1(\Omega, \mathbb{R}^m)\}. \tag{1.11}$$

For $d < N$, let us define $N^* = \frac{N}{N-d}$. It is known that

$$\nabla^d B(\Omega) \hookrightarrow L^q(\Omega), \forall\, q \leqq N^*, \tag{1.12}$$

$$\nabla^d B(\Omega) \underset{C}{\hookrightarrow} L^q(\Omega), \forall\, q < N^{*}\,[1]. \tag{1.13}$$

A particular case of Lemma I.1 in P.-L. LIONS [13] is

Lemma 1.3. *Let (u_n) be a bounded sequence in $\nabla^d B(\Omega)$, converging weakly to some u and such that $|\nabla^d u_n|$ converges weakly to μ and $|u_n|^{N^*}$ converges tightly to ν where μ, ν are bounded non-negative measures on Ω. Then there is a positive constant c, a sequence (x_j) of distinct points in Ω and a sequence (ν_j) of non-negative numbers such that*

$$\nu = |u|^{N^*} + \sum_{j \in J} \nu_j\, \delta_{x_j}, \tag{1.14}$$

$$\mu \geqq |\nabla^m u| + c \sum_{j \in N} \nu_j^{1/N^*}\, \delta_{xj} \tag{1.15}$$

where δ_{xj} denotes the Dirac measure at x_j.

We need another lemma which is also a particular case of a result in [13].

Lemma 1.4. *Let μ, ν be two bounded non-negative measures on $\mathbb{R}^N$ satisfying for some constant $c_0 > 0$*

$$\left(\int_{\mathbb{R}^N} |\varphi|^{N^*}\, d\nu\right)^{1/N^*} \leqq c_0 \left(\int |\varphi|\, \mu\right) \tag{1.16}$$

for all φ in $\mathcal{D}(\Omega)$. Then there are a positive constant c_0, a sequence (x_j) of distinct points in Ω and a sequence ν_j of non-negative numbers such that

$$\nu = \Sigma\, \nu_j\, \delta_{xj}, \qquad \mu \geqq c_0^{-1}\, \Sigma\, \nu_j^{1/N^*}\, \delta_{xj}. \tag{1.17}$$

Given a differential operator S with constant or $\mathscr{C}^\infty$ coefficients, we define

$$X = \{u \in L^1(\Omega, \mathbb{R}^N), Su \in M^1(\Omega, E)\}$$

where E denotes the space of symmetric tensors of order two in $\mathbb{R}^N$. X is a Banach space under the natural norm

$$|u|_X = |u|_1 + |Su|_T \tag{1.19}$$

[1] $X \underset{C}{\hookrightarrow} Y$ means that the injection from X into Y is compact.

where $| \ |_1$ denotes the L^1 norm and $| \ |_T$ denotes the total variation. We may also endow X with its weak topology which we define by the following family of semi-distances:

$$e(u, v, \varphi) = |u - v|_1 + \left|\int (Su - Sv)\,\varphi\right| \tag{1.20}$$

where φ belongs to $c_0(\Omega)$. Thus u_m tends to u weakly in Y if

$$u_m \to u \quad \text{in } L^1(\Omega),$$
$$Su_m \to Su \quad \text{in } M^1(\Omega) \text{ vaguely.}$$

I.2. Compactness when S is the gradient or the strain tensor operator

In this section we discuss conjectures on the space X introduced in Section I.1 for the case when S is either the gradient ∇^d with $d < N$ or the strain tensor operator ε defined by

$$\varepsilon_{ij}(T) = \tfrac{1}{2}\left[\frac{\partial T_i}{\partial x_j} + \frac{\partial T_j}{\partial x_i}\right].$$

We now give the first result of this section.

Theorem 1.1. *Let X be the space (1.18) when S is either ∇^d for $d < N$ or $S = \varepsilon$. We let $N^* = \dfrac{N}{N-d}$ if $S = \nabla^d$, $N^* = \dfrac{N}{N-1}$ if $S = \varepsilon$. Assume ψ is a proper lower semi-continuous convex function that satisfies* (1.1), (1.2). *Then from any bounded sequence (u_m) in X we may extract a subsequence, again denoted by (u_m), such that*

$$\begin{aligned} u_m &\to u \quad \text{in } L^q(\Omega), \quad \forall\, q < N^* \\ |u_m|^{N^*} &\to \nu \\ \psi(Su_m) &\to \mu \end{aligned} \tag{1.21}$$

where ν and μ are two bounded measures on Ω such that there exist a positive constant c_0, a sequence (x_j) of distinct points in Ω and a sequence (ν_j) of nonnegative numbers such that

$$\begin{aligned} \nu &= |u|^N + \sum_{j\in J} \nu_j \delta_{x_j}, \\ \mu &\geqq \psi(Su) + c_0 \sum_{j\in J} \nu_j^{1/N^*} \delta_{x_j}. \end{aligned} \tag{1.22}$$

Proof of Theorem 1.1. This proof borrows ideas from P. L. Lions. In either case, $X \hookrightarrow L^{N^*}$ and $X \underset{c}{\hookrightarrow} L^q$, $q < N^*$; then if u_m is bounded in X, using (1.9), we easily obtain (1.21) for a subsequence. It remains to prove (1.22). Using Lemma 1.3 we obtain

$$\nu = |u|^{N^*} + \Sigma\, \nu_j\, \delta_j,$$
$$|Su_m| \to \mu^* \geqq |Su| + \Sigma\, \nu_j^{1/N^*}\, \delta_{x_j}.$$

Now using Lemma 1.1, we find

$$\mu \geqq \psi(Su)$$

and using inequality (1.9),

$$\psi(Su_m) \geqq c_0(|Su_m| - 1).$$

We thus obtain

$$\mu \geqq c_0(\mu^* - 1).$$

Now let us assume $S = \nabla^d$. Since $\psi(Su)$ is absolutely continuous with respect to $|Su|$, to complete the proof it suffices to show that $|Su|$ and Dirac masses are orthogonal for $N \geqq 2$: indeed let $X_0 \in \Omega$ (we may suppose $X_0 = 0$) and let us show that $\lim\limits_{r\to 0} \int\limits_{B(0,r)} |Su| = 0$.

We introduce the notation $B^+(0,r) = \{(x', x_N), x_N > 0\} \cap B(0,r)$, $B^-(0,r) = \{(x', x_N), x_N < 0\} \cap B(0,r)$, $\Gamma = (x', 0) \cap B(0,r)$, where $B(0,r)$ denotes the ball of center 0 and radius r. We have

$$(Su)|_{x_N} = 0 = \left[\frac{\partial^{N-1}u}{\partial x_N^{N-1}}\right]_{x_N=0}$$

and $\left[\frac{\partial^{N-1}u}{\partial x_N^{N-1}}\right]$ belongs to $L^1(X_N = 0)$. Then for given $\delta > 0$ we may choose $r > 0$ such that

$$\left|\int_\Gamma \left|\left[\frac{\partial^{N-1}u}{\partial x_N^{N-1}}\right]\right| |dx'| < \delta.\right.$$

Moreover, Su is a bounded measure on $\Omega \cap \{X_N > 0\}$ and on $\Omega \cap \{X_N < 0\}$. Thus we may choose $r > 0$ such that

$$\int\limits_{B(0,r)} |Su| = \int\limits_{B^+(0,r)} |Su| + \int\limits_{B^-(0,r)} |Su| + \int\limits_{B(0,r)\cap\{x_N=0\}} \left|\left[\frac{\partial^{N-1}u}{\partial x_N^{N-1}}\right]\right| \leqq 3\delta,$$

which shows indeed that $|Su|$ and the Dirac masses are orthogonal for $N \geqq 2$.

We now turn to the case $S = \varepsilon$. Using a result of BRÉZIS & LIEB [4] and taking $v_m = u_m - u$, we see that

$$-\int |v_m|^{N^*} + \int |u_m|^{N^*} \to \int |u|^{N^*}$$

and we wish to show first that

$$|v_m|^{N^*} \to \Sigma\, \nu_j\, \delta_{x_j} = \lambda.$$

To that end we will show that for a subsequence we have

$$|v_m|^{N^*} \to \lambda, \qquad |\varepsilon(v_m)| \to \gamma,$$

where (λ, γ) are two bounded measures which satisfy the assumptions of Lemma 1.4.

Let φ be in $\mathscr{D}(\Omega)$, $\varphi \geqq 0$. We have by Proposition 1.3 of R. TÉMAM [24], for instance,

$$\left(\int |v_m\varphi|\right)^{1/N^*} \leqq c \int |\varepsilon(v_m\varphi)|.$$

For some positive constant c, by passing to the limit and using $u_m \to u$ in L^1 strongly, we get

$$(\int |\varphi|^{N^*} d\lambda)^{1/N^*} \leqq c(\int |\varphi| \nu) + 0$$

which implies

$$\nu = \sum_j \nu_j \, \delta_{x_j} \quad \text{for some } x_j \text{ and } \nu_j$$

and then

$$\lambda = |u|^{N^*} + \sum_j \nu_j \, \delta_{x_j}.$$

Now for φ in $\mathscr{D}(\Omega)$ we have

$$\begin{aligned}(\int |u_m \varphi|^{N^*}) &\leqq c \int \varepsilon \, |u_m \varphi| \\ &\leqq c(\int \varepsilon \, |u_m \varphi| + \int |u_m \otimes \nabla \varphi|) \\ &\leqq c_1 \int \psi(\varepsilon(u_m)) \, \varphi + c_2 \int \varphi + \int |u_m \otimes \nabla \varphi|\end{aligned}$$

and then passing to the limit,

$$(\int |\varphi|^{N^*} d\nu) \leqq c_1 \int \varphi \, d\mu + c_2 \, (\int |u \otimes \nabla \varphi| + \int \varphi).$$

Now let ϱ be in $\mathscr{D}(]-1, +1[^N)$, $\varrho = 1$ on a neighborhood of 0, and $\eta > 0$; let $\varrho_i(x) = \varrho\left(\frac{x - x_i}{\eta}\right)$. We have

$$\lim_{\eta \to 0} \int |\varrho_i|^{N^*} d\nu = v_i, \quad \lim_{\eta \to 0} \int \varrho_j = 0$$

and

$$\begin{aligned}\int u \otimes \nabla \varrho_i &\leqq \left(\int_{B(X_i, \eta)} |u|^{N^*}\right)^{1/N^*} \frac{1}{\eta} \left[\int \left|\nabla \varrho \left(\frac{x - x_i}{\eta}\right)^N dx\right|\right]^{1/N} \\ &= \left(\int_{B(X_i, \eta)} |u|^{N^*}\right)^{1/N^*} \frac{1}{\eta} \left[\int \eta^N \, |\nabla \varrho(u)|^N \, du\right]^{1/N} \\ &= \left(\int_{B(X_i, \eta)} |u|^{N^*}\right)^{1/N^*} |\nabla \varrho|_{L^N}.\end{aligned}$$

Since u belongs to L^{N^*}, this last expression tends to zero when $\eta \to 0$. We finally obtain

$$\mu \geqq \frac{1}{c} \nu_i^{N^*} \delta_{x_i}$$

or every x_i, $i \in J$, and

$$\mu \geqq \frac{1}{c} \Sigma \, \nu_i^{N^*} \delta_{x_i}.$$

Now, as with ∇^d, we see that for $N \geqq 2$, $\int_{\{x_0\}} |\varepsilon(u)| = 0$ and using

$$\psi(\varepsilon(u)) \leqq \mu = \underline{\lim} \, \psi(\varepsilon(u_m)),$$

we obtain

$$\mu \geqq \psi(\varepsilon\,|u|) + \frac{1}{c}\Sigma\, \nu_i^{N^*}\,\delta_x\,,$$

which completes the proof of Theorem 1.1. □

Theorem 1.2. *If (u_m) is bounded in X and $\psi(Su_m)$ tends to a measure μ which has no Dirac masses, then $u_m \to u$ in $L^{N^*}_{\mathrm{loc}}(\Omega)$. If, moreover, the convergence of $\psi(Su_m)$ towards μ is tight, then $u_m \to u$ in $L^{N^*}(\Omega)$. For $S = \varepsilon$, or ∇, we have in addition, $\gamma_0 u_m \to \gamma_0 u$ in $L^1(\Gamma)$.*

Proof. Let us suppose that $\psi(Su_n)$ tends to a measure μ which has no atoms. Then by (1.21) $\nu_j = 0$ for all $j \in \mathbb{N}$ and $|u_n|^{N^*}$ converges vaguely to $|u|^{N^*}$. We note that for all open Ω' relatively compact in Ω,

$$\int_{\Omega'} |u|^{N^*} \to \int_{\Omega'} |u|^{N^*}.$$

Indeed, we have by lower semi-continuity of the total variation on open sets

$$\int_{\Omega'} |u|^{N^*} \leqq \varliminf \int_{\Omega'} |u_n|^{N^*},$$

and by upper semi-continuity on compact sets

$$\varlimsup \int_{\Omega'} |u_n|^{N^*} = \varlimsup \int_{\overline{\Omega'}} |u_n|^{N^*} \leqq \int_{\overline{\Omega'}} |u|^{N^*}.$$

These two inequalities imply

$$\lim \int_{\Omega'} |u_n|^{N^*} = \int_{\Omega'} |u|^{N^*}.$$

This property and the weak* convergence in L^{N^*} are sufficient to ensure the strong convergence in $L^{N^*}(\Omega')$. Let us now suppose that $\psi(Su_m)$ tends tightly to $\psi(Su)$ on Ω, and show that $u_m \to u$ in $L^{N^*}(\Omega)$. Suppose for a while that this is true when $S = \nabla$ or ε. Then, if $d > 1$ and $\nabla^d u_m$ is bounded in $M^1(\Omega, E)$, $\nabla^{d-1} u_m$ is bounded in BV. The result for $d = 1$ implies that $\nabla^{d-1} u_m$ tends to $\nabla^{d-1} u$ in $L^{N/N-1}$, then $u_m \to u$ in $W^{d-1, N/N-1}$, and finally $u_m \to u$ in $L^{N/N-d}$. Assume now that $S = \nabla$ or ε. Use of Lemma A.4 in the appendix of [26], Part 4, provides for every $\varepsilon > 0$ given a compact set K in Ω such that

$$\int_{\Omega\setminus K} \psi(Su_n) < \varepsilon \left(\int_{\Omega\setminus K} |Su_n| < \frac{1}{c_1}(\varepsilon)\right)$$

for all $n \in \mathbb{N}$. For given $\delta > 0$ we define the open set

$$\Omega_\delta = \{x \in \Omega,\, d(x, \partial\Omega) < \delta\}$$

and put $\Gamma_\delta = \partial\Omega_\delta - \partial\Omega$. We may choose $\delta > 0$ such that $\Omega_\delta \subset \Omega \setminus K$, $\int_{\Omega_\delta} \psi(Su) < \varepsilon$ and $\int_{\Omega_\delta} |u|^{N^*} < \varepsilon$. By Fubini's theorem, u_m tends to u in $L^1(\Gamma_\delta)$ for almost every δ, so there exists $N \in \mathbb{N}$ such that for $m > N$, $\int_{\Gamma_\delta} |u_m - u| < \varepsilon$.

We now define $\tilde{u}_m$ to be equal to u_m in Ω_δ and to u in $\Omega \setminus \Omega_\delta$. $\tilde{u}_m$ belongs to $X(\Omega)$ and

$$\begin{aligned} S\tilde{u}_m &= Su_m/\Omega_\delta + (u_m - u)\,\vec{n}\;\partial_{\Gamma_\delta} \quad \text{if } S = \nabla, \\ &= Su_m/\Omega_\delta + \mathscr{C}(u_m - u)\;\partial_{\Gamma_\delta} \quad \text{if } S = \varepsilon \end{aligned}$$

where δ_{Γ_δ} denotes the superficial $(N-1)$-dimensional measure on Γ_δ, and $\mathscr{T}(p)$ is the tensor with components $\mathscr{T}_{ij}(p) = \frac{1}{2}(p_i n_j + p_j n_i)$. This leads to the estimate

$$\begin{aligned} \int_\Omega S(\tilde{u}_m - u) &\leqq \int_{\Omega_\delta} |S(u_m - u)| + \int_{\Gamma_\delta} |u_m - u| \\ &\leqq \int_{\Omega_\delta} |Su_m| + \int_{\Omega_\delta} |Su| + \int_{\Gamma_\delta} |u_m - u| \leqq 3\varepsilon, \end{aligned}$$

and thus

$$\begin{aligned} \int_{\Omega\setminus\Omega_\delta} |u_m - u|^{N^*} &\leqq \int_\Omega |\tilde{u}_m - u|^{N^*} \\ &\leqq c(\Omega) \int_\Omega |\tilde{u}_m - u| \\ &\leqq c(\Omega) \cdot 3\varepsilon. \end{aligned}$$

All these inequalities imply that $\int_{\Omega\setminus\Omega_\delta} |u_m|^{N^*} \leqq c(\Omega')\,\varepsilon$ and using Lemma (A.4) of [26] Part 4, we finally conclude that $|u_m|^{N^*}$ converges tightly to $|u|^{N^*}$.

We wish now to prove that if $S = \varepsilon$ or ∇ and u_m tends to u in $L^1(\Omega)$, $\psi(Su_m) \to \psi(Su)$ tightly on Ω, then

$$u_m \to u \text{ in } L^1(\partial\Omega).\,^1$$

Let us remark to begin with that according to Lemma A.4 in the appendix of [26], Part 4, for every $\eta > 0$ there is a compact set K in Ω such that for every $m \in \mathbb{N}$, $\int_{\Omega\setminus K} \psi(Su_m) < \eta$. This implies by (1.2) that

$$\int_{\Omega\setminus K} |Su_m| < C_0\eta + \operatorname{meas}(\Omega \setminus K). \tag{1.22}$$

For $\delta > 0$ we define once more the open set $\Omega_\delta = \{x \in \Omega, d(x, \partial\Omega) < \delta\}$ and set $\Gamma_\delta = \delta\Omega_\delta \setminus \partial\Omega$. We may choose $\delta > 0$ such that $\Omega_\delta \subset \Omega \setminus K$ and $\int_{\Omega_\delta} |Su| < \eta$. By Fubini's theorem, $u_m \to u$ in $L^1(\Gamma_\delta)$, for almost every δ, so we may choose $M \in \mathbb{N}$, such that for $m \geqq M$

$$\int_{\Gamma_\delta} |u_m - u| < \eta, \tag{1.24}$$

$$\int_\Omega |u_m - u| < \eta \tag{1.25}$$

because u_m converges to u in $L^1(\Omega)$. We now recall the following lemma, proved in the appendix of [26].

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[1] This property has been proved by R. TÉMAM [22] for $\psi = |\cdot|$.

Lemma A.1. *There is a positive constant c such that for every* $F \in L^\infty(\partial\Omega)$ *(or* $\vec{F} \in L^\infty(\partial\Omega, \mathbb{R}^N)$*) there is a* $\sigma \in L^\infty(\Omega, \mathbb{R}^N)$, $\operatorname{div} \sigma \in L^\infty(\Omega)$ *(or* $\sigma \in L^\infty(\Omega, E)$, $\operatorname{div} \sigma \in L^\infty(\Omega, \mathbb{R}^N)$*) which satisfies* $\sigma \cdot n = F$ *(or* $\vec{\sigma} \cdot n = \vec{F}$*), and*

$$(|\sigma|_\infty + |\operatorname{div} \sigma|_\infty) \leqq c\,|F|_\infty. \tag{1.26}$$

We apply the above lemma with $F_m = \dfrac{u_m - u}{|u_m - u|}$. Let us then consider σ_m which satisfies $\sigma_m \cdot n = F_m$ and

$$|\sigma_m|_\infty + |\operatorname{div} \sigma_m|_\infty \leqq c. \tag{1.27}$$

We have by Green's formula on Ω:

$$\int_\Gamma F_m \cdot (u_m - u) - \int_{\Gamma_\delta} (\sigma_m \cdot n)(u_m - u)$$
$$= \int_{\Omega_\delta} S(u_m - u) : \sigma_m + \int_{\Omega_\delta} (u_m - u) \operatorname{div} \sigma_n$$

and then using (1.28), for $m \geqq M$, we deduce

$$\int_\Gamma |u_m - u| \leqq c \int_{\Gamma_\delta} |u_m - u| + \int_{\Omega_\delta} |Su| + c \int_{\Omega_\delta} |u_m - u| \leqq 2c\varepsilon + 2\varepsilon$$

by virtue of (1.24)–(1.27). □

I.3. Compactness when $S = \nabla^N$ in $\mathbb{R}^N$

We present here certain theorems on compactness in the space $\nabla^N B(\Omega)$ (where N is the dimension of Ω). The space $\nabla^2 B(\Omega)$ for $N = 2$ has been denoted by HB(Ω) and has been studied in [7]. It is easy to show that the following results established in [7], when $N = 2$, are valid if $N > 2$, namely,

$$\nabla^N B(\Omega) \hookrightarrow C(\Omega) \tag{1.28}$$

and if $\partial\Omega$ is sufficiently smooth (piecewise C^N for instance),

$$\nabla^N B(\Omega) \hookrightarrow C(\bar{\Omega}). \tag{1.29}$$

We may endow $\nabla^N B(\Omega)$ with a weak topology which can be defined by the countable family of semi-distances

$$e(u_n, u, \varphi) = |u_n - u| + \left|\int \nabla^N(u_n - u)\,\varphi\right| \tag{1.30}$$

where φ belongs to $\mathscr{C}_0(\Omega)$.

If we endow the space of continuous functions on Ω with the topology induced by pointwise convergence, the injection $(\nabla^N B(\Omega), e) \hookrightarrow C(\Omega)$ is not compact, as it is shown by the following example:

Let $\varphi \in \mathscr{D}(\Omega)$, with $\varphi = 1$ on a neighborhood of $0 \in \Omega$ and let φ_n be defined as

$$\varphi_n(x) = \varphi(nx).$$

It is easy to see that φ_n is bounded in $\nabla^N B(\Omega)$. Indeed

$$\nabla^N \varphi_n(x) = n^N (\nabla^N \varphi)(nx)$$

and then

$$\int_{\mathbb{R}^N} |\nabla^N \varphi_n|(x)\, dx = \int n^N |\nabla^N \varphi|(nx)\, dx = \int |\nabla^N \varphi|(x)\, dx.$$

This property and the convergence of φ to 0 in $L^1(\Omega)$ imply that φ_n tends to zero in $\nabla^N B(\Omega)$ weakly. But $\varphi_n(0) = \varphi(0) = 1$ shows that $\varphi_n(0) \not\to 0$.

For $\nabla^N B(\Omega)$ equipped with the topology of the norm it is clear that the injection of $\nabla^N B(\Omega)$ in $C(\Omega)$ is compact. A more useful topology introduced in [7], is the following:

Let ψ be a convex function which satisfies (1.1). We define the distance

$$d_\psi(u_1, u_2) = |u_1 - u_2|_1 + \left|\int \psi(\nabla^N u_1) - \psi(\nabla^N u_2)\right| \tag{1.31}$$

and state

Theorem 1.3. *Let $X = \nabla^N B(\Omega)$, and let (u_m) be a bounded sequence in X. Then we can find a subsequence, still denoted (u_m), such that $u_m \to u$ in $L^q(\Omega)$ $\forall\, q < +\infty$ and in $\nabla^N B(\Omega)$ weakly, where u belongs to $\nabla^N B(\Omega)$. Moreover there are a positive constant c, a sequence (x_j) of distinct points in Ω and a sequence v_j of non-negative numbers such that*

$$u_m \to u + \sum_{j \in \mathbb{N}} v_j \chi_j \quad \textit{pointwise on } \Omega, \tag{1.33}$$

$$\psi(\nabla^N u_m) \to \mu \geqq \psi(\nabla^N u) + c \sum_j |v_j|\, \delta_{x_j}. \tag{1.34}$$

Corollary 1.1. *If $u_m \to u$ in $\mathrm{HB}(\Omega)$ weakly and if $|\nabla\nabla u_m|$ tends to a measure μ which has no atom, then*

$$u_m(x) \to u(x) \qquad \forall\, x \in \Omega.$$

Remark 1.2. Corollary 1.1 proves a conjecture of De Giorgi [5].

Proof of Theorem 1.3. We may suppose that u_m tends to u in $\nabla^N B(\Omega)$ weakly and that $\psi(\nabla^N u_m)$ and $|\nabla^N(u_m - u)|$ converges vaguely to bounded measures μ and $\tilde{\mu}$. We denote by $\bar{v}$ the function defined for every x in Ω by $\bar{v}(x) = \overline{\lim}\,(u_m - u)(x) = \overline{\lim}\,(v_m(x))$. $\bar{v}$ is bounded and $\bar{v}$ equals zero almost everywhere with respect to the Lebesgue measure dx.

Let x_0 be such that $|\bar{v}(x_0)| > 0$. We define $\varrho_\varepsilon(x) = \varrho\left(\dfrac{x - x_0}{\varepsilon}\right)$ where ϱ is as in the proof of Theorem 1.1 and $\varepsilon < d(x_0, \partial\Omega)$. Then

$$\begin{aligned} |(\bar{v}_m \varrho)(x_0)| &\leqq c \int_\Omega |\nabla^N(\bar{v}_m \varrho)| \\ &\leqq c \int_\Omega |\nabla^N v_m|\, |\varrho| + \sum_{p<N} \int_\Omega |C_N^p \nabla^p v_m\, \nabla^{N-p} \varrho| \end{aligned}$$

where c is some positive constant. For $p < N$, $\nabla^p v_m \to 0$ in $L^1(\Omega)$ and then

$$|v(x_0)| = |(v\varrho)(x_0)| \leqq c \int |\varrho| \, d\tilde{\mu}.$$

Letting ε tend to zero, we conclude that $\tilde{\mu}\{x_0\} > 0$; thus the set of x such that $|\bar{v}(x)| > 0$ is at most countable. We may write

$$\bar{v} = \Sigma \bar{v}_j 1_{\{x_j\}}$$

where $\bar{v}_j = \bar{v}(x_j) \geqq 0$. By the procedure carried through for $\underline{v} = \underline{\lim}\,(u_m - u)$, we obtain

$$\underline{v} = \Sigma \underline{v}_j 1_{\{y_j\}}$$

for some $\underline{v}_j$ in $\mathbb{R}^+$, and y_j in Ω. Of course we may replace (x_j) and (y_j) by their union and reindex in order to have $\bar{v} = \Sigma \bar{v}_j 1_{\{x_j\}}$, $\underline{v} = \Sigma \underline{v}_j 1_{\{x_j\}}$. Let us now note that

$$\tilde{\mu} \leqq \mu + |\nabla^N u|$$

and apply μ to ϱ_i, $\varrho_i(x) = \varrho\left(\dfrac{x - x_i}{\varepsilon}\right)$. Letting ε tend to 0 we obtain

$$\sup\,(|\bar{v}_i|, |\underline{v}_i|)\, \delta_{x_i} \leqq c\mu,$$

because $\int |\nabla^N u| \varrho_i \to 0$ when $\varepsilon \to 0$. (Indeed, according to a theorem of [7], $|\nabla^N u|$ has no Dirac masses for $N \geqq 2$).

To finish the proof of Theorem 1.3, let $(u_{m'})$ be a subsequence of (u_m) such that

$$\psi(\nabla^N u_{m'}) \to \mu$$

and let J be the subset of $\mathbb{N}$ such that $\mu(\{x_j\}] > 0$ for all $j \in J$. For $x \neq x_j$ the inequality, above, implies that $u_{m'}(x) - u(x) \to 0$. Now, by the diagonal process, we may extract from $(u_{m'})$ a subsequence denoted $(u_{m''})$ such that

$$(u_{m''} - u)(x_i) \to v_i \in [\underline{v}_i, \bar{v}_i],$$

for all $j \in J$. We thus obtain

$$u_{m''} \to u + \Sigma v_i 1_{\{x_i\}}$$

with

$$\psi(Su_{m''}) \to \tilde{\mu} \geqq \psi(Su) + c\,\Sigma\,|v_i|\,\delta_{x_i}. \quad \square$$

We now obtain a sharper result.

Theorem 1.4. *If (ξ_m) is weakly convergent to ξ in $\nabla^N B(\Omega)$, and if $|\xi_{m,1,2,\ldots,N}|$ [1] convergents vaguely to μ such that*

$$\mu\{\vec{x} \in \mathbb{R}^N, x_i = y_i\} = 0 \quad \text{for all } \vec{y} \text{ in } \mathbb{R}^N.$$

Then (ξ_n) converges uniformly to ξ on every compact subset of Ω.

[1] $\xi_{,i_1 i_2 \ldots i_k}$ denotes the partial derivative $\dfrac{\partial^k \xi}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}$.

Proof. We assume that $N=2$; the extension to the general case is straightforward. By hypothesis $|\mu|\{(x_1,x_2)\}=0\ \forall\,(x_1,x_2)=x\in\Omega$, so by Theorem 1.3, (ξ_n) converges pointwise to ξ in Ω. To that end we show that (ξ_n) is equicontinuous and then use the Ascoli-Arzelà theorem. Let $x=(x_1,x_2)$ be in Ω and ϱ in $\mathscr{D}(\Omega)$, $\varrho=1$ in a neighborhood of x. Noticing that $|(\varrho\xi_n)_{,12}|$ tends tightly to a measure which has no mass on lines parallel to the coordinate axes, we see that it suffices to prove the theorem for a sequence (ξ_n) with fixed compact support in Ω which converges tightly to a measure μ which has no mass on lines $x_i=$ const. This implies that $\xi_{n,12}$ is also tightly convergent to μ on the open set: $\Omega_1=]-\infty,x_1[\times]-\infty,x_2[\cap\Omega$ of $\mathbb{R}^N$ as well as on its complement Ω_2. Consequently, using Lemma (A.4), for every given $\varepsilon>0$, we perceive that there are compact subsets K_1 of Ω_1 and K_2 of Ω_2 such that

$$\int_{\Omega_1\setminus K_1}|\xi_{n,12}|<\varepsilon,\qquad \int_{\Omega_2\setminus K_2}|\xi_{n,12}|<\varepsilon. \tag{1.36}$$

Now let δ be a positive number such that $\delta<\inf(d(K_1,\partial\Omega_1),d(K_2,\partial\Omega_2))$ and let $y=(y_1,y_2)$ be such that $|x_1-y_1|<\delta$, $|x_2-y_2|<\delta$. We define the sets

$$B(x_1,x_2)=\,]\inf(x_1,x_2),\sup(x_1,x_2)[\times\mathbb{R},$$

$$B(y_1,y_2)=\mathbb{R}\times]\inf(y_1,y_2),\sup(y_1,y_2)[.$$

By using (1.36) and the assumptions on δ, we obtain

$$\int_{(B(x_1,x_2)\cup B(y_1,y_2))\cap\Omega}|\xi_{n,12}|<2\varepsilon.$$

Therefore, employing the explicit expression of $\xi_n(x,y)$ and $\xi_n(z,t)$ given in [7], we get

$$\begin{aligned}|\xi_n(x,y)-\xi_n(z,t)|&=\left|\int_{-\infty}^{x}\int_{-\infty}^{y}\xi_{n,12}-\int_{-\infty}^{z}\int_{-\infty}^{t}\xi_{n,12}\right|\\&\leqq\int_{(B(x,y)\cup B(y,t))\cap\Omega}|\xi_{n,12}|<2\varepsilon\end{aligned}$$

which implies that the sequence (ξ_n) is equicontinuous. $\square$

It is natural to ask whether the convergence is uniform on $\bar{\Omega}$, when $\partial\Omega$ is sufficiently regular. The following theorem establishes uniform convergence in $\bar{\Omega}$ in a sufficiently general case.

Definition. *Ω has the square cone property if it can be covered with a finite collection $\{(O_i),\ i\in I\}$ of open sets with the following property: for each i there is an open cone C_i of right angle and vertex 0 such that*

$$(x+C_i)\cap O_i\subset\Omega,\quad \textit{for all } x \textit{ in } \cap\,\bar{\Omega}\cap O_i.$$

Proposition 1.1. *Let us suppose Ω has the square cone property. Then if (ξ_n) converges weakly to ξ in $\nabla^N B$ and $|\xi_{n,12}|$ converges tightly to a measure μ which has no mass on lines parallels to the coordinate axes, then (ξ_n) converges uniformly to ξ in $\bar{\Omega}$.*

Proof. We assume once again that $N = 2$. To begin with, we note that the computation of $\xi(x)$ in terms of $\xi_{,12}$ given in [7] may be extended to the x's in $\partial\Omega$ when Ω has the square cone property: indeed suppose $x \in \partial\Omega$ and let $(\vec{e}_1, \vec{e}_2)$ be an orthonormal basis of $\mathbb{R}^2$ such that C is the cone

$$C = \{\lambda e_1 + \mu e_2, \lambda \leqq 0, \mu \leqq 0\}$$

with $((x_1, x_2) + C) \cap B \subset \Omega$, where B is a bounded set centered at $(x_1, x_2) = x$. By choosing ϱ in $\mathscr{D}(B)$, $\varrho = 1$ on (x_1, x_2), it is easy to show, following the proof of Theorem 3.3 of [7], that

$$\eta(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} (\varrho\xi)_{,12}(t, \lambda)\, dt\, d\lambda \tag{1.38}$$

is continuous on $B \cap \Omega$ and coincides with $\varrho\xi$ for almost every $y = (y_1, y_2)$ in Ω.

The remainder of the proof is merely a straightforward modification of the arguments used in the proof of Theorem 2.4. $\square$

Corollary 1.2. *Let Ω verify* (1.37) *and assume that* $|\xi_{n,12}| \to |\xi_{,12}|$ *tightly on Ω. Then ξ_n converges uniformly to ξ on $\bar{\Omega}$.*

Proof. We saw in [7] that if

$$\Omega = \Omega_1 \cup \Omega_2, \quad \bar{\Omega}_1 \cap \bar{\Omega}_2 = \Gamma, \quad \Omega_1 \cap \Omega_2 = \emptyset$$

where Γ is $(N-1)$-dimensional and C^2 and $\vec{\eta}$ is the unit normal pointing from Ω_1 to Ω_2, we have for ξ in $\mathrm{HB}(\Omega)$

$$\nabla\nabla\xi_{/\Omega} = \nabla\nabla\xi_{/\Omega_1} + \nabla\nabla\xi_{/\Omega_2} + \left(\frac{\partial\xi^2}{\partial n} - \frac{\partial\xi^1}{\partial n}\right)(n \otimes n)\,\delta_\Gamma.$$

This implies that if Γ is a line $x_i = \text{const.}$ then $\vec{n} = e_j$ for $j \neq i$ and

$$(\nabla\nabla\xi)_{|\Gamma} = \left(\frac{\partial\xi}{\partial x_j}\right) e_i \otimes e_i\, \delta_\Gamma$$

hence $|\xi_{,12}|_{/\Gamma} \equiv 0$. The assertion of the corollary now follows by Proposition 1.1. $\square$

We showed in [7] that when $\partial\Omega$ is piecewise C^2, the trace of $u \in \mathrm{HB}(\Omega)$ on $\partial\Omega$, which is defined since u belongs to $\mathscr{C}(\bar{\Omega})$, is in fact contained in a much smaller space, denoted by $\gamma_0(W^{2,1}(\Omega))$. We also defined the second trace map γ_1 and showed that $\gamma_1(W^{2,1}(\Omega)) = L^1(\Gamma)$. When (u_m) is weakly convergent to u in $\mathrm{HB}(\Omega)$, it is not generally true that $\gamma_0 u_m \to \gamma_0 u$ in $\gamma_0(W^{2,1})$, nor that $\gamma_1 u_m \to \gamma_1 u$ in $L^1(\Gamma)$. The proposition below gives conditions sufficient that $\gamma_0 u_n \mapsto \gamma_0 u$.

Proposition 1.2. *Let us assume $\psi(\nabla\nabla u_n)$ converges tightly to $\mu \in M^1(\Omega)$ and $u_n \to u$ in* HB(Ω) *weakly. Then $u_n \to u$ in $\gamma_0(W^{2,1})$ (strongly), and $\gamma_1 u_n \to \gamma_1 u$ in $L^1(\Gamma)$.*

Proof. Since $\psi(\nabla\nabla u_n)$ converges tightly, by Lemma A.4, we may choose for all $\varepsilon > 0$, a positive number δ such that if $\Omega_\delta = \{x \in \Omega, d(x, \partial\Omega) < \delta\}$, then

$$\int_{\Omega_\delta} |\nabla\nabla u_n| \leqq c_1 \int_{\Omega_\delta} \psi(\nabla\nabla u_n) + c_2 \operatorname{meas} \Omega_\delta \leqq c\varepsilon, \tag{1.39}$$

for all $n \in \mathbb{N}$. We may impose, in addition, the assumption

$$\int_{\Omega_\delta} |\nabla\nabla u| < \varepsilon$$

and if $\Gamma_\delta = \partial\Omega_\delta - \partial\Omega$, we may suppose that $u_n \to u$ in $W^{2,1}(\Gamma_\delta)$, and $\dfrac{\partial u_n}{\partial n} \to \dfrac{\partial u}{\partial n}$ in $L^1(\Gamma)$. Thus there is an N_1 such that for all $n > N$,

$$|u_n - u|_{W^{2,1}(\Gamma_\delta)} < \varepsilon, \qquad \left|\frac{\partial}{\partial \vec{n}}(u_n - u)\right|_{L^1(\Gamma_\delta)} < \varepsilon. \tag{1.40}$$

Now recalling the definition of $|u_n - u|_{\gamma_0(W^{2,1})}$ we let $F_n \in \gamma_0(W^{2,1}(\Omega))^{*}$ [1] and $G_m \in L^\infty(\Gamma)$ be such that

$$|u_n - u|_{\gamma_0(W^{2,1})} \leqq \langle F_n, u_n - u\rangle + \varepsilon, \left|\frac{\partial}{\partial \vec{n}}(u_n - u)\right|_{L^1(\Gamma)} \leqq \langle G_n, u_n - u\rangle + \varepsilon \tag{1.41}$$

and

$$|F_n|_{\gamma_0(W^{2,1})^*} \leqq 1, \qquad |G_n|_\infty \leqq 1.$$

Now we use Lemma A.2 in the Appendix of [26] to provide a sequence M_n in $L^\infty(\Omega, E)^1$, with $\nabla \cdot \nabla M_n \in L^\infty(\Omega)$, such as to satisfy $b_0(M_n) = F_{n|\Gamma}$, $b_1(M_n) = G_{n|\Gamma}$, and $|M_n|_\infty + |\nabla \cdot \nabla \cdot M_n|_\infty \leqq c\{|F_n|\, \gamma_0(W^{2,1})^* + |G_n|_{L^\infty(\Gamma)}\} \leqq 2c$.

By the generalized Green's formula (of [6]) we obtain

$$\langle F_m, u_n - u\rangle - \int_\Gamma \frac{\partial}{\partial \vec{n}}(u_n - u)\, G_n - \langle b_0(M_n), u_n - u\rangle_{\Gamma_\delta} + \langle b_1(M_n)\, u_n - u\rangle_{\Gamma_\delta}$$
$$= \int_{\Omega_\delta} \nabla\nabla(u_n - u)\, M_n - \int_{\Omega_\delta} (u_n - u)\, \nabla \cdot \nabla \cdot M_n$$

---

[1] We denote by $\gamma_0(W^{2,1})^*$ the dual space of $\gamma_0(W^{2,1})$ endowed with the induced topology of $W^{2,1}$, i.e. $|u|_{\gamma_0(W^{2,1})} = \inf_{v = u/\Gamma} \|v\|_{W^{2,1}}$.

[2] For every M in $L^\infty(\Omega, E)$, $\nabla \cdot \nabla \cdot M \in L^\infty(\Omega)$, we defined in [6] $b_0(M)$ and $b_1(M)$ as elements of $\gamma_0(W^{2,1})^* \times L^\infty(\Gamma)$. They coincide with $\operatorname{div}(M \cdot n) + \dfrac{\partial}{\partial s}(M \cdot n \cdot t)$ and $M \cdot n \cdot n$ when M is sufficiently regular, and are such that the following Green's formula holds:

$$\int \nabla\nabla u : M - \nabla \cdot \nabla \cdot M u = \int b_1(M) \frac{\partial u}{\partial n} - \langle b_0(M), u\rangle,$$

for every u in $W^{2,1}(\Omega)$.

and then, using (1.39) $\to$ (1.41) and the convergence in $L^1(\Omega)$, we get for some N and for every $n > N$

$$\left|\frac{\partial}{\partial \vec{n}}(u_n - u)\right|_{L^1(\Gamma)} + |u_n - u|_{\gamma_0(W^{2,1})^*} \leqq \int (u_n - u) F_n$$

$$\leqq c\left(\int_{\Omega_\delta} |\nabla\nabla u_n| + \int_{\Omega_\delta} |\nabla\nabla u| + c \int_{\Omega_\delta} |u_n - u| + \left|\frac{\partial}{\partial n}(u_n - u)\right|_{L^1(\Gamma_\delta)}\right.$$

$$\left. + \|u_n - u\|_{W^{2,1}(\Gamma_\delta)}\right) \leqq 4c\varepsilon$$

which finishes the proof of Proposition 1.2. $\square$

Part II. Applications

2.1. Applications to the Calculus of Variations, Capillarity Theory and Antiplane Shear in Plasticity

The theory of minimal surfaces, capillarity and antiplane shear in plasticity lead to variational problems having the common form

$$\inf_{v = u_0/\Gamma_0} \left(\int_\Omega \{\phi(\nabla v) - Lv\}\right) \tag{2.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, with a smooth boundary Γ, $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, Γ_0 and Γ_1 are two open connected subsets of Γ_1, $Lv = \int_\Omega fv + \int_{\Gamma_1} Fv, f \in L^N(\Omega)$, $F \in L^\infty(\Gamma_1)$, $u_0 \in H^{1/2}(\Gamma)$, and ϕ is a convex function on $\mathbb{R}^N$ which satisfies (1.1) and (1.2). In [23] TÉMAM showed that the dual problem of (2.1) is simply

$$\sup_{\substack{\operatorname{div} p + f = 0 \\ p \cdot n = g/\Gamma_1}} \left(-\int_\Omega \phi^*(p) + \int_{\Gamma_1} p \cdot n u_0\right). \tag{2.2}$$

Moreover, $\inf(2.1) = \sup(2.2)$. The existence of a minimizer for (2.1) has been studied in [23], [21]. It is natural to consider the space $\mathrm{BV}(\Omega) = \{u \in L^1(\Omega), \nabla u \in M^1(\Omega, \mathbb{R}^N)\}$, and to extend Problem (2.1) to a relaxed problem defined by

$$\inf_{u \in \mathrm{BV}(\Omega)} \left(\int_\Omega \phi(\nabla u) + \int_{\Gamma_0} \phi_\infty((u_0 - u)\vec{n}) - Lu\right) \tag{2.3}$$

where ϕ_∞ denotes the asymptotic function of ϕ, or, equivalently,

$$\phi_\infty(\xi) = \lim_{t \to +\infty} \frac{\phi(t\xi)}{t} = \chi_K^*(\xi).$$

Moreover, it has been shown that $\inf(2.1) = \inf(2.3)$. If we replace in (2.3) or (2.1) L by λL, the analysis in [23] shows that $\inf(2.3) > -\infty$ if and only if $\mathcal{K} \cap \mathcal{S}_\lambda \neq \emptyset$, where

$$\mathcal{S}_\lambda = \{\sigma \in L^2, \operatorname{div}\sigma + \lambda f = 0, \sigma \cdot n = \lambda F/\Gamma_1\}$$

and

$$\mathscr{K} = \{\sigma \in L^2(\Omega, \mathbb{R}^N), \sigma(x) \in K \text{ a.e.}\}.$$

In order to determine the set of σ's such that $\operatorname{div}\sigma + \lambda f = 0$, $\sigma \cdot n = \lambda F$, TÉMAM introduced in [23] the limit analysis problem

$$\inf_{\substack{v \in H^1(\Omega) \\ v=0|\Gamma_0 \\ Lv=1}} \int_\Omega \phi_\infty(\nabla v) \tag{2.4}$$

and studied its dual. He obtained that

$$\inf_{\substack{v \in H^1(\Omega) \\ v=0|\Gamma_0 \\ Lv=1}} \int_\Omega \phi_\infty(\nabla v) = \sup_{\substack{\exists \sigma \in K \\ \operatorname{div}\sigma + \lambda f = 0 \\ \sigma \cdot \nu = \lambda F}} \{\lambda\} = \bar{\lambda}. \tag{2.5}$$

In an analogous fashion one may define the problem

$$\inf_{\substack{v \in H^1(\Omega) \\ v=0|\Gamma_0 \\ Lv=-1}} \int_\Omega \phi_\infty(\nabla v) \tag{2.6}$$

and prove that

$$\inf_{\substack{v \in H^1(\Omega) \\ v=0|\Gamma_0 \\ Lv=-1}} \int \phi_\infty(\nabla v) = \sup_{\substack{\exists \sigma \in K \\ \operatorname{div}\sigma + \lambda f = 0 \\ \sigma \cdot \nu = \lambda F}} \{-\lambda\} = \underline{\lambda}. \tag{2.7}$$

Therefore a condition necessary and sufficient that $\mathscr{K} \cap \mathscr{S}_\lambda$ not be empty is

$$-\underline{\lambda} \leqq \lambda \leqq \bar{\lambda}.$$

Moreover it may be shown, as in [23], [21], that if $-\underline{\lambda} < \lambda < \bar{\lambda}$, all the minimizing sequences of Problem (2.1) or (2.3) with L replaced by λL, are bounded in $\mathrm{BV}(\Omega)$. This hypothesis (called the *safe load condition*) is sufficient to show that a generalized solution for (2.3) exists. In what follows, we use Theorems 1.1 and 1.2 of the first chapter to improve this existence theorem: for instance, the minimizing sequences converge in a topology stronger than the weak topology of $\mathrm{BV}(\Omega)$. Another more interesting application concerns the limit analysis problem: we will show that, unless $\bar{\lambda}$ is a special number $\bar{\bar{\lambda}}$ which will be determined later, the relaxed problem of limit analysis

$$\inf_{\substack{v \in \mathrm{BV}(\Omega) \\ Lv=1}} \left\{ \int_\Omega \phi_\infty(\nabla v) + \int_{\Omega_0} \phi_\infty(-v)\,\vec{\eta} \right\} \tag{2.8}$$

has a solution in $\mathrm{BV}(\Omega)$.

We begin by giving a new formulation of Problem (2.8): Let Ω' be a bounded open set of $\mathbb{R}^N$ such that $\Omega' \cap \Omega = \emptyset$, $\bar{\Omega}' \cap \bar{\Omega} = \bar{\Gamma}_0$ and define $\Omega_0 = \Omega$

$\cup \Omega' \cup \Gamma_0$. Assume (for technical reasons) that $u_0 = \chi_0 u_0$, where χ_0 denotes the characteristic function of Γ_0, and let U_0 be in $H^1(\Omega')$ with $U_0 \mid \Gamma_0 = u_0$; we may extend $u \in \mathrm{BV}(\Omega)$ to Ω_0 by setting

$$\tilde{u} = \begin{cases} u \text{ in } \Omega \\ U_0 \text{ in } \Omega' \end{cases}. \tag{2.10}$$

Then $\tilde{u}$ belongs to $\mathrm{BV}(\Omega_0)$, and applying Green's formula for $\mathrm{BV}(\Omega_0)$ we obtain

$$\nabla \tilde{u} = \nabla u_{|\Omega} + (U_0 - u)\, \vec{n}\, \delta_{\Gamma_0} + \nabla u_{0|\Omega'}. \tag{2.11}$$

Let us now recall certain results obtained by R. KOHN & R. TÉMAM [12].

Proposition 2.1. *For any u in $\mathrm{BV}(\Omega)$ and any σ in $L^N(\Omega, \mathbb{R}^N)$ with $\operatorname{div} \sigma \in L^N(\Omega)$, let $(\sigma, \nabla u)$ denote the distribution on Ω defined for $\varphi \in \mathscr{C}_0^\infty(\Omega)$ by*

$$\int_\Omega (\sigma, \nabla u)\, \varphi = - \int_\Omega \operatorname{div} \sigma u \varphi - \int_\Omega \sigma \nabla \varphi u. \tag{2.12}$$

Then $(\sigma, \nabla u)$ may be extended as a bounded measure on Ω which is absolutely continuous with respect to $|\nabla u|$ and satisfies

$$|(\sigma \cdot \nabla u)| \leqq |\sigma|_\infty\, |\nabla u|. \tag{2.13}$$

Moreover Green's formula holds in the form

$$\int_\Omega (\sigma \cdot \nabla u) \varphi + \int_\Omega u \operatorname{div} \sigma \varphi + \int_\Omega u \sigma \nabla \varphi = \int_\Gamma \sigma \cdot n u \varphi \tag{2.14}$$

for each φ in $\mathscr{C}^1(\overline{\Omega})$.

Let us now define on $\Omega \cup \Gamma_0$ the measure

$$\nabla \tilde{u} \cdot \tau = \nabla u \cdot \tau_{/\Omega} + \tau \cdot n(u_0 - u)\, \delta_\Gamma.$$

By applying Green's formula (2.14) to the pair (u, τ) where $u \in \mathrm{BV}(\Omega)$, $\tau \in L^N(\Omega, E) \cap \mathscr{S}_{ad}$, we get

$$\begin{aligned} \int_{\Omega \cup \Gamma_1} \nabla \tilde{u} \cdot \tau &= \int_\Omega \nabla \tilde{u} \cdot \tau + \int_{\Gamma_1} \tau \cdot n(u_0 - u) \\ &= - \int_\Omega u \operatorname{div} \tau + \int_{\Gamma_1} \tau \cdot n u_0 + \int_{\Gamma_0} \tau \cdot n u \\ &= \int_\Omega f u + \int_{\Gamma_1} F u + \int_{\Gamma_1} \tau \cdot n u_0 \end{aligned}$$

which implies that we may write Problem (2.8) in the following form:

$$\inf_{\substack{\tilde{u} \in \mathrm{BV}(\Omega_0) \\ u = u_{0|\Omega'}}} \left\{ \int_\Omega \psi(\nabla \tilde{u}) - \int_\Omega \nabla \tilde{u} \cdot \tau + \int_{\Gamma_0} \tau \cdot n u_0 \right\}. \tag{2.15}$$

We now state the main result of this subsection.

Theorem 2.1. *Assume there is a minimizing sequence (u_m) for Problem* (2.3) *that is bounded in* $\mathrm{BV}(\Omega_0)$. *Then we may extract from it a subsequence, likewise denoted by (u_m), such that $\tilde{u}_m$ tends to u, $u \in \mathrm{BV}(\Omega)$, in the following sense*:

$$\tilde{u}_m \to u \quad \textit{in } L^1(\Omega_0),$$
$$\nabla \tilde{u}_m \to \nabla u \quad \textit{vaguely in } \Omega_0,$$
$$\psi(\nabla \tilde{u}_m) \to \psi(\nabla u) \quad \textit{vaguely in } \Omega_0,$$
$$u_m \to u \quad \textit{in } L^{N/N-1}_{\mathrm{loc}}(\Omega_0).$$

Moreover u is a solution of Problem (2.15).

Proof. Let (u_m) be a minimizing sequence which is bounded in $\mathrm{BV}(\Omega_0)$; then there is a subsequence, likewise denoted by (u_m), such that

$$\tilde{u}_m \to \tilde{u} \quad \text{in } L^1(\Omega_0),$$
$$\nabla \tilde{u}_m \to \nabla \tilde{u} \quad \text{in } M^1(\Omega_0) \text{ vaguely}.$$

We may suppose, in addition, that $\psi(\nabla \tilde{u}_m)$ converges vaguely to a bounded measure μ on Ω_0 that satisfies $\psi(\nabla \tilde{u}) \geqq \mu$, by virtue of a lemma in [8]. Now let δ be a positive number. We denote by Ω_0 the open set $\Omega_\delta = \{x \in \mathbb{R}^N, d(x, \Gamma_1) < \delta\}$.

Let φ_δ be in $\mathscr{D}(\Omega_\delta)$ with $\varphi_\delta = 1$ in a neighborhood of Γ_1 in $\mathbb{R}^N$ and set $g_\delta = 1 - \varphi_\delta$. For every τ in $L^N(\Omega, K) \cap \mathscr{S}_{ad}$ we have, by applying Green's formula (2.14),

$$\int_{\Omega \cup \Gamma_0} (\nabla(\tilde{u}_m) : \tau)\, g_\delta = - \int_\Omega \tilde{u}_m \operatorname{div} \tau g_\delta - \int_\Omega u_m \tau \nabla g_\delta + \int_{\Gamma_0} u_0 \tau \cdot n g_\delta .$$

The right-hand side converges to $-\int_\Omega \tilde{u} \operatorname{div} \tau g_\delta - \int_\Omega u \tau \nabla g_\delta + \int_{\Gamma_0} u_0 \tau \cdot n g_\delta$, which equals $\int_{\Omega \cup \Gamma_0} (\nabla \tilde{u} : \tau)\, g_\delta$ (by virtue of Green's formula (2.14)). Because $\mu = \psi(\nabla u_0)$ on Ω', the weak lower semi-continuity of the integral on open sets of bounded measure implies that

$$\begin{aligned}
\int_{\Omega \cup \Gamma_0} \mu g_\delta &= \int_{\Gamma_0} \mu g_\delta - \int_{\Omega'} \psi(\nabla u_0)\, g_\delta \\
&\leqq \underline{\lim} \int_{\Omega_0} (\nabla \tilde{u}_m)\, g_\delta - \int_{\Omega'} \psi(\nabla U_0)\, g_\delta \\
&= \lim \int_{\Omega \cup \Gamma_0} \psi(\nabla \tilde{u}_m)\, g_\delta .
\end{aligned}$$

Now, for given $\varepsilon > 0$ we may choose $\delta > 0$ sufficiently small that

$$\int_{\operatorname{supp} \varphi_\delta} \psi^*(\tau) < \varepsilon, \tag{2.16}$$

$$\int_\Omega |\mu|\, \varphi_\delta < \varepsilon. \tag{2.17}$$

$$|\tau|_\infty \int_\Omega |\nabla \tilde{u}|\, g_\delta < \varepsilon. \tag{2.18}$$

According to Lemma (7.8) of [24], the inequality

$$\psi(\nabla \tilde{u}_m) - \nabla \tilde{u}_m : \tau \geqq -\psi^*(\tau) \tag{2.19}$$

holds on $\Omega \cap \Gamma_0$, and so

$$\begin{aligned}
\int_{\Omega\cup\Gamma_0} \psi(\nabla\tilde{u}) - \int_{\Omega\cup\Gamma_0} (\nabla\tilde{u}:\tau) &\leqq \int_{\Omega\cup\Gamma_0} \mu - \int_{\Omega\cup\Gamma_0} \nabla\tilde{u}:\tau \\
&= \int_{\Omega\cup\Gamma_0} \mu g_\delta - \int_{\Omega\cup\Gamma_0} (\nabla\tilde{u}:\tau)\, g_\delta + \int_{\Omega\cup\Gamma_0} \mu\varphi_\delta - \int_{\Omega\cup\Gamma_0} (\nabla\tilde{u}:\tau)\,\varphi_\delta \\
&\leqq \underline{\lim} \int_{\Omega\cup\Gamma_0} (\psi(\nabla\tilde{u}_m) - (\nabla\tilde{u}_m:\tau))\, g_\delta + \int_{\Omega\cup\Gamma_0} |\mu|\,\varphi_\delta + \int_{\Omega_0} |\tau|_\infty\, |\nabla\tilde{u}|\,\varphi_\delta \\
&\leqq \underline{\lim} \int_{\Omega\cup\Gamma_0} (\psi(\nabla\tilde{u}_m) - (\nabla\tilde{u}_m:\tau))\, g_\delta + 2\varepsilon \quad \text{(by (2.17) and (2.18))} \\
&\leqq \underline{\lim} \int_{\Omega\cup\Gamma_0} (\psi(\nabla\tilde{u}_m) - (\nabla\tilde{u}:\tau))\,(g_\delta + \varphi_\delta) + 2\varepsilon + \int_{\Omega_0} \psi^*(\tau)\, g_\delta \quad \text{(by (2.19))} \\
&\leqq \underline{\lim} \int_{\Omega\cup\Gamma_0} (\psi(\nabla\tilde{u}_m) - (\nabla\tilde{u}_m:\tau)) + 2\varepsilon + \varepsilon \quad \text{(by use of (2.16))}.
\end{aligned}$$

Since ε is arbitrary, we obtain

$$\begin{aligned}
\int_{\Omega\cup\Gamma_0} \psi(\nabla\tilde{u}) - \nabla\tilde{u}:\tau &\leqq \int_{\Omega\cup\Gamma_0} \mu - \int_{\Omega\cup\Gamma_0} \nabla\tilde{u}:\tau \\
&= \inf P - \int_{\Gamma_0} \tau\cdot n u_0 \\
&\leqq \int_{\Omega\cup\Gamma_0} \psi(\nabla\tilde{u}) - \int_{\Omega\cup\Gamma_0} \nabla\tilde{u}:\tau
\end{aligned}$$

which implies both that u is a solution for Problem (2.3) and that $\psi(\nabla\tilde{u}) = \mu$. It suffices then to apply Theorem 1.1 in Part I to conclude that $u_m \to u$ in $L^{N/N-1}_{\text{loc}}(\Omega_0)$.

We now present a sharper form of Theorem 2.1.

Theorem 2.2. *Let us suppose that there are a* $\delta > 0$ *and a* $\sigma \in L^N\left(\Omega, \frac{K}{1+\delta}\right)$ *such that* $\operatorname{div}\sigma \in L^N(\Omega)$, $\sigma\cdot n = F/\Gamma_1$. *Moreover, assume that there is a minimizing sequence* (u_m) *for Problem* (2.15) *that is bounded in* BV. *Then we may extract from* (u_m) *a subsequence, likewise denoted by* (u_m), *such that*

$$u_m \to u \quad \text{in } L^{N/N-1}(\Omega_0),$$

$$u_{m/\Gamma_1} \to u/\Gamma_1 \quad \text{in } L^1(\Gamma_1).$$

Remark 2.1. Theorem 2.2 improves Theorem 2.1 mainly in showing $u_{m/\Gamma_1} \to u/\Gamma_1$ because, in general, weak convergence in $\mathrm{BV}(\Omega_0)$ does not imply convergence of the traces on Γ. Let us now see an important consequence of Theorem (2.2). We consider the problem

$$\inf_{\substack{u\in \mathrm{BV}(\Omega_0)\\ u=u_{0/\Omega'}}} \left\{ \int_{\Omega\cup\Gamma_0} (\psi(\nabla\tilde{u})) - \lambda \int_\Omega fu - \lambda \int_{\Gamma_1} Fu \right\} \tag{2.20}$$

and we define the real numbers

$$\bar{\bar{\lambda}} = \sup_{\substack{\exists\sigma\in L^N(\Omega,K),\operatorname{div}\sigma\in L^N\\ \sigma\cdot n=\lambda F/\Gamma_1}} \{\lambda\}, \tag{2.21}$$

$$\underline{\underline{\lambda}} = \sup_{\substack{\exists\sigma\in L^N(\Omega,K),\operatorname{div}\sigma\in L^N\\ \sigma\cdot n=\lambda F/\Gamma_1}} \{-\lambda\}. \tag{2.22}$$

It is obvious that $\bar{\lambda} \leqq \bar{\bar{\lambda}}$, $-\underline{\lambda} \geqq -\underline{\underline{\lambda}}$. It may easily be seen that the strict inequality $\lambda < \bar{\bar{\lambda}}$ implies that there exist $\delta > 0$ and σ in $L^N\left(\Omega, \frac{K}{1+\delta}\right)$ such that $\operatorname{div}\sigma \in L^N$ and $\sigma \cdot n = \lambda F/\Gamma_1$. We then obtain

Corollary 2.1. *If* $0 \leqq \lambda \leqq \bar{\lambda} < \bar{\bar{\lambda}}$ *(respectively* $-\underline{\underline{\lambda}} < -\underline{\lambda} \leqq \lambda \leqq 0$*) and if there is a minimizing sequence* (u_m) *for Problem* (2.15) *(or* (2.8)*) which is bounded in* $\mathrm{BV}(\Omega_0)$*, then*

$$\tilde{u}_m \to \tilde{u} \quad in \quad \mathrm{BV}(\Omega_0) \quad weakly,$$

$$\psi(\nabla \tilde{u}_m) \to \psi(\nabla \tilde{u}) \quad tightly\ on\ \Omega_0$$

and

$$u_{m/\Gamma_1} \to u/\Gamma_1. \tag{2.25}$$

Moreover u *is a solution of Problem* (2.8).

Remark 2.2. When $\lambda \in]-\underline{\lambda}, \bar{\lambda}[$, it was shown in [24] that every minimizing sequence of (2.8) is bounded in $\mathrm{BV}(\Omega_0)$. Moreover, in that case we may give a rather simple proof of Corollary 2.1, which is due to R. TÉMAM [25]. Indeed, let λ' be in $]\lambda, \bar{\lambda}[$. We may show (following the proof of Theorem 2.1) that if (u_m) is a minimizing sequence which converges weakly to u in $\mathrm{BV}(\Omega_0)$, then

$$\int_{\Omega\cup\Gamma_0} \psi(\nabla \tilde{u}) - \lambda' L u \leqq \underline{\lim} \int_{\Omega\cup\Gamma_0} \psi(\nabla u_m) - \lambda' L u_m.$$

Since (u_m) is a minimizing sequence and u is admissible for Problem (2.8),

$$\int_{\Omega\cup\Gamma_0} \psi(\nabla u_m) - \lambda L u_m \to \int_{\Omega\cup\Gamma_0} \psi(\nabla u) - \lambda L u.$$

Now, writing

$$\int_{\Omega\cap\Gamma_0} \psi(\nabla u) - \lambda L u$$

$$= \frac{\lambda}{\lambda'}\left[\int_{\Omega\cup\Gamma_0} \psi(\nabla u) - \lambda' L u\right] + \left(1 - \frac{\lambda}{\lambda'}\right) \int_{\Omega\cup\Gamma_0} \psi(\nabla u)$$

$$\leqq \frac{\lambda}{\lambda'} \underline{\lim} \left(\int_{\Omega\cup\Gamma_0} \psi(\nabla \tilde{u}_m) - \lambda' L u_m\right) + \left(1 - \frac{\lambda}{\lambda'}\right) \int_{\Omega\cup\Gamma_0} \psi(\nabla \tilde{u}_m)$$

we show that $\lim_{m\to+\infty}\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u}_m)=\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u}_m)$, which implies that $\psi(\nabla\tilde{u}_m)\to\psi(\nabla u)$ tightly on Ω_0. By applying Theorem 1.2, we get $\tilde{u}_{m/\partial\Omega_0}\to u/_{\partial\Omega_0}$, which means that $u_{m/\Gamma_1}\to u/\Gamma_1$. □

Let $\tau=(1+\delta)\,\sigma\in\mathscr{K}$. We may show, as in the proof of Theorem 2.1, that if (u_m) is a minimizing sequence which tends weakly to u then

$$\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u})-\int_{\Omega\cup\Gamma_0}\nabla\tilde{u}:\tau\leqq\underline{\lim}\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u}_m)-\int_{\Omega\cup\Gamma_0}\nabla\tilde{u}_m:\tau.$$

Now let $\bar{\sigma}$ be the solution of the dual problem of (2.15). According to Lemma (7.8) of [24], $\psi(\nabla\tilde{u}_m)-\nabla\tilde{u}_m:\bar{\sigma}\geqq-\psi^*(\bar{\sigma})$ on $\Omega\cup\Gamma_0$, and since (u_m) is a minimizing sequence,

$$\int_{\Omega\cup\Gamma_0}\psi(\nabla(\tilde{u}_m))-\nabla(\tilde{u}_m)\,\bar{\sigma}+\psi^*(\bar{\sigma})\to 0.$$

Thus

$$\psi(\nabla(u_m))-\nabla(u_m):\bar{\sigma}+\psi^*(\bar{\sigma})\to 0\quad\text{ tightly on }\Omega\cup\Gamma_0.$$

Moreover, Theorem 2.1 asserts that u is a generalized solution, *i.e.*

$$\psi(\nabla\tilde{u})-\nabla\tilde{u}:\bar{\sigma}=-\psi^*(\bar{\sigma})\quad\text{ on }\Omega\cup\Gamma_0.$$

Using the fact that $\int_\Omega(u_m-u)\operatorname{div}(\bar{\sigma}-\tau)\to 0$ and Green's formula (2.14) on $\Omega\cup\Gamma_0$ we get

$$\begin{gathered}\int_{\Gamma_1}(\bar{\sigma}-\tau)\cdot nu\leqq\underline{\lim}\int_{\Gamma_1}(\bar{\sigma}-\tau)\cdot nu_m,\\ -\delta\int_{\Gamma_1}Fu\leqq-\underline{\lim}\,\delta\int_{\Gamma_1}Fu_m.\end{gathered}\tag{2.26}$$

The opposite inequality is valid without the above assumption on σ. Indeed

$$\begin{aligned}\int_{\Omega\cup\Gamma_0}\nabla\tilde{u}:\bar{\sigma}&=\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u})+\int_\Omega\psi^*(\bar{\sigma})\\&\leqq\underline{\lim}\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u}_m)+\int_\Omega\psi^*(\bar{\sigma})\\&=\underline{\lim}\int_{\Omega\cup\Gamma_0}\nabla\tilde{u}_m:\sigma.\end{aligned}$$

Using again $\lim_m\int_\Omega(u-u_m)\operatorname{div}\bar{\sigma}=0$, we obtain $\int_{\Gamma_1}Fu\leqq\underline{\lim}\int_{\Gamma_1}Fu_m$ so by (2.26)

$$\int_{\Gamma_1}Fu=\lim\int_{\Gamma_1}Fu_m.$$

Now using once again the equality

$$\begin{aligned}\lim\int_{\Omega\cup\Gamma_0}&\psi(\nabla\tilde{u}_m)-\int_\Omega fu_m-\int_{\Gamma_1}Fu_m\\&=\int_{\Omega\cup\Gamma_0}\psi(\nabla\tilde{u})-\int_\Omega fu-\int_{\Gamma_1}Fu\end{aligned}$$

we obtain

$$\lim \int_{\Omega\cup\Gamma_0} \psi(\nabla\tilde{u}_m) = \int_{\Omega\cup\Gamma_0} \psi(\nabla u).$$

Then $\psi(\nabla\tilde{u}_m)$ tends to $\psi(\nabla u)$ tightly on Ω_0, and Theorem 1.2 in Part I ensures that $(\tilde{u}_m)$ tends to u in $L^{N/N-1}(\Omega_0)$ and (u_m) tends to u in $L^1(\Gamma_1)$.

Let us now see how the conclusions in Part I may be used to interpret limit analysis for previous problems. Let us recall the statement of the relaxed limit analysis problem introduced in Part II, (2.8),

$$\inf_{\substack{u\in \mathrm{BV}(\Omega)\\ Lu=1}} \left\{ \int_\Omega \phi_\infty(\nabla u) + \int_{\Gamma_0} \phi_\infty(-u)\vec{n} \right\}, \tag{2.28}$$

which may be written also as

$$\inf_{\substack{\tilde{u}\in \mathrm{BV}(\Omega_0)\\ \tilde{u}=0/\Omega'\\ Lu=1}} \left\{ \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla\tilde{u}) \right\}. \tag{2.29}$$

This problem has the dual

$$\sup_{\substack{\exists\sigma\in L^N(\Omega,K),\operatorname{div}\sigma+\lambda f\in L^N\\ \sigma\cdot n=\lambda F/\Gamma_1}} \{\lambda\} = \bar{\lambda}.$$

It is clear that $\bar{\lambda} \leqq \bar{\bar{\lambda}}$ but it is not generally true that $\bar{\lambda} = \bar{\bar{\lambda}}$ as may be seen from the following example. Assume that $N=1$, $K=\{x, |x|\leqq 1\}$, $\Omega =]0,1[$, f is defined as

$$f = \begin{cases} 1 & \text{on } [0, 1/2[\\ -1 & \text{on }]1/2, 1[\end{cases}$$

and $F_1 = F_2 = 0$. These data imply that $\bar{\bar{\lambda}} = +\infty$. Let us suppose now that $\sigma\in K$, $\sigma' + \lambda f = 0$, $\sigma(1) = 0$. We easily get $\sigma(t) = -\lambda \int_0^t f(u)\,du$. We must have $1 \geqq \sup|\sigma(t)| = |\sigma(1/2)| = \lambda/2$ and then

$$\bar{\lambda} = 2.$$

We now state the existence of solutions of the relaxed analysis problem.

Theorem 2.3. *Assume* $\bar{\lambda} < \bar{\bar{\lambda}}$. *Then Problem* (2.29) *admits a minimizer* u. *More precisely if* $(\tilde{u}_m)$ *is a minimizing sequence for Problem* (2.29) *we may extract from it a subsequence still denoted* $(\tilde{u}_m)$ *such that*

$$\tilde{u}_m \to u \quad \textit{in } \mathrm{BV}(\Omega_0),$$

$$\phi_\infty(\nabla\tilde{u}_m) \to \phi_\infty(\nabla u) \quad \textit{tightly on } \Omega_0,$$

and consequently $u_m \to u$ *in* $L^1(\Gamma_1)$, $Lu = 1$, *and* $u_m \to u$ *in* $L^{N/N-1}(\Omega)$.

Corollary 2.2. *If K is the convex set* $K = \left\{\sigma, \sum_i |\sigma_i|^2 \leqq 1\right\}$ *and if* $\bar{\lambda} < \frac{1}{|F|_\infty}$, *the limit analysis Problem* (2.29) *has a solution u in* BV(Ω) [1].

Proof of Theorem 2.3. The proof is very similar to that of Theorem 2.2. Indeed let $(\tilde{u}_m)$ be a minimizing sequence for Problem (2.28). Then $(\tilde{u}_m)$ is bounded in $\mathrm{BV}(\Omega_0)$ (since $\phi_\infty(\nabla(\tilde{u}_m))$ is bounded) and $Lu_m = 1$. We may extract from $(\tilde{u}_m)$ a subsequence, still denoted by $(\tilde{u}_m)$, such that $\tilde{u}_m$ tends to u in $\mathrm{BV}(\Omega_0)$ weakly. Of course $u = 0$ in Ω'. If $\bar{\sigma}$ is a solution for the dual problem of (2.28), following the argument in the proof of Theorem (2.2), we get

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla u) - \int_{\Omega\cup\Gamma_0} \nabla u : \bar{\sigma} &\leqq \lim \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla \tilde{u}_m) - \nabla \tilde{u}_m : \bar{\sigma} \\ &= \bar{\lambda} - \bar{\lambda}\, Lu_m \\ &= 0.\end{aligned}$$

This implies, as in the proof of Theorems 2.1 and 2.2, that $\phi_\infty(\nabla u) = \nabla u : \bar{\sigma}$ on $\Omega \cup \Gamma_0$, so we immediately see that if $\nabla u \neq 0$, $Lu = \int_{\Omega\cup\Gamma_0} \nabla u : \bar{\sigma} = \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla u) \neq 0$, and since φ_∞ is homogeneous, that u/Lu is a solution of (2.29). In fact we will show that under the assumption $\bar{\lambda} < \bar{\bar{\lambda}}$, we cannot have $u \equiv 0$, because $Lu = \lim Lu_m = 1$. To that end, we proceed as in the proof of Theorem 2.2. Since $\bar{\lambda} < \bar{\bar{\lambda}}$, let λ be in $]\bar{\lambda}, \bar{\bar{\lambda}}[$ and τ be in $L^N(\Omega)$, with $\tau(x) \in K$ almost everywhere, with respect to the Lebesgue measure dx. Since τ belongs to K, $\phi_\infty(\nabla u) - \nabla u : \tau \geqq -\phi^*_\infty(\tau) = 0$, and by lower semi-continuity of the total variation of the positive measure $\phi_\infty(\nabla u) - \nabla u : \tau$ on the open set Ω_0, we deduce

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla u) - \int_{\Omega\cup\Gamma_0} \nabla u : \tau &\leqq \underline{\lim} \int_{\Omega\cup\Gamma_0} (\phi_\infty(\nabla \tilde{u}_m) - \nabla \tilde{u}_m : \tau) \\ &= \underline{\lim} \int_{\Omega\cup\Gamma_0} \nabla \tilde{u}_m : \bar{\sigma} - \int_{\Omega\cup\Gamma_0} \nabla \tilde{u}_m : \tau.\end{aligned}$$

This implies, by virtue of $\phi_\infty(\nabla u) = \nabla u : \bar{\sigma}$ on $\Omega \cup \Gamma_0$, that

$$\int_\Omega \nabla u : (\bar{\sigma} - \tau) \leqq \underline{\lim} \int_\Omega \nabla \tilde{u}_m : (\bar{\sigma} - \tau),$$

so by Green's formula (2.14),

$$\int_{\Gamma_1} Fu \geqq \underline{\lim} \int_{\Gamma_1} Fu_m.$$

On the other hand

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \nabla u : \bar{\sigma} &= \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla u) \\ &\leqq \lim \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla \tilde{u}_m) \\ &= \bar{\lambda}\, Lu_m \\ &= \int_{\Omega\cup\Gamma_0} \nabla u_m : \bar{\sigma}\end{aligned}$$

[1] This Corollary proves a conjecture of R. V. KOHN [11].

and then $\int_{\Gamma_1} Fu \leqq \underline{\lim} \int_{\Gamma_1} Fu_m$. We have thus obtained that

$$\lim_{m\to+\infty} \int_{\Gamma_1} Fu_m = \int_{\Gamma_1} Fu$$

and so

$$1 = Lu_m \to Lu .$$

Returning to the preceding inequalities, we find, in addition, that

$$\lim_{m\to+\infty} \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla\tilde{u}_m) = \int_{\Omega\cup\Gamma_0} \phi_\infty(\nabla u),$$

which means that $\phi_\infty(\nabla\tilde{u}_m)$ converges tightly to $\phi_\infty(\nabla u)$ on Ω_0, and that $u_{m/\Gamma_1} \to u/\Gamma_1$, by virtue of Theorem 1.2. □

2.2. Application to perfect plasticity (Hencky's Law)

Let us consider an elastic-plastic material which occupies an open bounded set Ω in $\mathbb{R}^N$, $N \geqq 3$, and is subjected to a body force of density f, and to a traction F on an open connected part Γ_1 of $\partial\Omega$. The displacement u is required to equal u_0 on Γ_0 (where Γ_0 is also open and $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$). The problem of determining the equilibrium configuration of the material is to find (u, σ) defined on $\mathbb{R}^N \times E$ such that

$$\begin{aligned} u &= u_{0/\Gamma_0} \\ \sigma^D &\in K^D \\ \operatorname{div}\sigma + F &= 0 \quad \text{in } \Omega \\ \sigma\cdot n &= F/\Gamma_1 \\ (\varepsilon(u) - A\sigma &: \tau - \sigma) \leqq 0 . \end{aligned} \tag{2.30}$$

Here E denotes the space of tensors of order two on $\mathbb{R}^N$, $\xi^D = \xi - \frac{1}{3}(\operatorname{tr}\xi)\,\delta_{ij}$ is the deviator of $\xi \in E$, K^D is a bounded convex set of E^D, which contains 0 in its interior, and A denotes the operator of linear elasticity for homogeneous and isotropic materials, *i.e.*,

$$A_{ijkh} = \frac{1}{9K_0}\delta_{ij}\delta_{kh} + \frac{1}{4\mu}\delta_{ik}\delta_{jh},$$

where K_0 and μ are positive constants (see [18]). If we set

$$\phi^*(\xi) = \begin{cases} \frac{1}{2}A\xi : \xi = \dfrac{1}{18K_0}(\operatorname{tr}\xi)^2 + \dfrac{1}{4\mu}|\xi^D|^2 & \text{if } \xi \in K = K^D \times \mathbb{R}I \\ +\infty \ \text{if } \xi \notin K, \end{cases} \tag{2.31}$$

the conjugate $\phi = \phi^{**}$ of ϕ^* is a convex lower semi-continuous and proper function which satisfies

$$\phi(\xi) = \frac{K_0}{2}(\operatorname{tr}\xi)^2 + \phi^D(\xi^D)$$

with

$$c_0(|\xi^D| - 1) \leqq \phi^D(\xi^D) \leqq c_1(|\xi^D| + 1), \tag{2.32}$$

for some positive constants c_0, c_1. The equations (2.30) may be formulated under one of the weaker variational forms[2]

$$\inf_{\substack{v\in H^1(\Omega) \text{ or } \mathrm{LD}(\Omega) \\ \operatorname{div} v\in L^2(\Omega) \\ v=u_0/\Gamma_0}} \left\{ \int_\Omega \phi^D(\varepsilon^D(v)) + \frac{K}{2}\int (\operatorname{div} v)^2 - \int_\Omega f\cdot v - \int_{\Gamma_1} gv \right\} \tag{2.33}$$

in terms of the displacement, and

$$\sup_{\substack{\operatorname{div}\sigma+F=0 \\ \sigma\cdot\nu=g/\Gamma_1 \\ \sigma^D\in K^D}} \left\{ -\int \phi^{*D}(\sigma^D) - \frac{1}{18K_0}\int_{\Gamma_0} \sigma\cdot\nu u_0 \right\}^1 \tag{2.34}$$

in terms of the stress, where $f\in L^N(\Omega)$, $F\in L^\infty(\Gamma_1)$, $u_0\in H^{1/2}(\Gamma_0)$, $u_0 = \chi_0 u_0$. These problems are dual, and inf (2.33) = sup (2.34) (*cf.* [24], [21]).

The study made in [24], [21] shows that if we replace in (2.33) L by λL, inf (2.33) $> -\infty$ if and only if $\mathscr{K} \cap \mathscr{S}_\lambda \neq \emptyset$, where $\mathscr{S}_\lambda$ and $\mathscr{K}$ are defined as

$$\mathscr{S}_\lambda = \{\sigma\in L^2(\Omega, E), \operatorname{div}\sigma + \lambda f = 0 \text{ in } \Omega,\ \sigma\cdot\nu = \lambda g/\Gamma_1\},$$

$$\mathscr{K} = \{\sigma\in L^2(\Omega, E), \sigma(x)\in K\,dx \text{ a.e.}\}.$$

In order to determine the λ's for which $\mathscr{K}\cap\mathscr{S}_\lambda \neq \emptyset$ we introduce, as in the previous section, the limit analysis problem

$$\inf_{\substack{Lv=1 \\ \operatorname{div} v=0 \\ v=0|\Gamma_0}} \int \phi_\infty(\varepsilon^D(v)), \tag{2.35}$$

the problem dual to which is

$$\sup_{\exists\sigma\in\mathscr{S}_\lambda\cup\mathscr{K}} \{\lambda\} = \bar{\lambda}. \tag{2.36}$$

The following equality holds:

$$\inf(2.39) = \sup(2.36).$$

In an analogous fashion, we may state the problem

$$\inf_{\substack{Lv=-1 \\ \operatorname{div} v=0 \\ v=0|\Gamma_0}} \int \phi_\infty(\varepsilon^D(v)). \tag{2.37}$$

We may define, similarly, $\underline{\lambda} = \sup_{\mathscr{S}_\lambda\cap\mathscr{K}=\emptyset} \{-\lambda\}$, and show that

$$\inf_{\substack{Lv=-1 \\ \operatorname{div} v=0 \\ v=0|\Gamma_0}} \int \phi_\infty(\varepsilon^D(v)) = \sup_{\mathscr{S}_\lambda\cap\mathscr{K}=\emptyset}\{-\lambda\} = \underline{\lambda}. \tag{2.38}$$

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[1] ν denotes the outer normal to Γ.

[2] $\mathrm{LD}(\Omega) = \{u\in L^1(\Omega, \mathbb{R}^N), \varepsilon(u)\in L^1(\Omega, E)\}$.

A necessary and sufficient condition for $\mathscr{S}_\lambda \cap \mathscr{K}$ not to be empty is that $-\underline{\lambda} \leqq \lambda \leqq \bar{\lambda}$. Moreover, if $-\underline{\lambda} < \lambda < \bar{\lambda}$ then every minimizing sequence of Problem (2.33), with L replaced by λL, is bounded in $\mathrm{BD}(\Omega)$. For that case KOHN & TÉMAM showed in [12] that a generalized solution for a relaxed form of (2.33) exists. Following the study made in the previous section, we strengthen this existence result (Theorem 2.4). Moreover, we will see that if $\bar{\lambda}$ is less than a positive number $\bar{\bar{\lambda}}$ which will be defined later, the relaxed problem of limit analysis, *viz.*,

$$\inf_{\substack{v\in \mathrm{BD}(\Omega)\\ Lv=1\\ v\cdot n=0|\Gamma_0\\ \operatorname{div} v=0}} \left(\int_\Omega \phi_\infty(\varepsilon^D(v)) + \int_{\Gamma_0} \phi_\infty(\mathscr{C}^D(-v))\right) \tag{2.40}$$ [1]

has a solution in $\mathrm{BD}(\Omega)$.

Now let us recall some of the results obtained in [12].

Proposition 2.2. *Assume that u is in* $\mathrm{BD}(\Omega)$, $\operatorname{div} u$ *is in* $L^2(\Omega)$ *and* σ *is in* $L^2(\Omega, E)$ *with* $\sigma^D \in L^\infty(\Omega, E)$ *and* $\operatorname{div}\sigma \in L^N(\Omega, \mathbb{R}^N)$. *Let* $\sigma^D\varepsilon^D(u)$ *denote the distribution on* Ω *defined for* φ *in* $\mathscr{C}_0^\infty(\Omega)$ *by*

$$\int_\Omega \sigma^D\varepsilon^D(u)\,\varphi = \int_\Omega \operatorname{div}\sigma u\varphi - \int_\Omega \sigma : u \otimes \nabla\varphi - \tfrac{1}{3}\int_\Omega \operatorname{tr}\sigma \operatorname{div} u .$$

Then $\sigma^D\varepsilon^D(u)$ *may be extended as a bounded measure on* Ω *which is absolutely continuous with respect to* $|\varepsilon^D(u)|$, *and the following weak version of Green's formula holds*:

$$\int_\Omega \sigma^D\varepsilon^D(u)\,\varphi + \tfrac{1}{3}\int_\Omega \operatorname{tr}\sigma \operatorname{div} u\varphi$$
$$= -\int_\Omega \operatorname{div}\sigma u\varphi - \int_\Omega \sigma u \otimes \nabla\varphi + \int_{\Gamma_1} \sigma\cdot n\cdot u\varphi$$

for each φ *in* $\mathscr{C}^1(\bar{\Omega})$.

We now define on $\Omega \cup \Gamma_0$ the following measure:

$$\varepsilon(\tilde{u}) : \sigma = \varepsilon(u) : \sigma/\Omega + \sigma^D \cdot n \cdot (u_0 - u)_t\, \delta_{\Gamma_0},$$ [2]

By applying Green's formula in Proposition 2.2 for σ in $L^2(\Omega, K) \cap \mathscr{S}_{ad}$, we get

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \varepsilon(\tilde{u}) : \sigma &= \int_\Omega \varepsilon(\tilde{u}) : \sigma + \int_{\Gamma_0} \sigma\cdot n\cdot(u_0 - u)\\ &= -\int_\Omega u \operatorname{div}\sigma + \int_{\Gamma_0} \sigma\cdot n\cdot u_0 + \int_{\Gamma_1} \sigma\cdot n\cdot u\\ &= \int_\Omega fu + \int_{\Gamma_1} Fu + \int_{\Gamma_0} \sigma\cdot n\cdot u_0 .\end{aligned}$$

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[1] $\mathscr{T}(p) = \dfrac{p_i\nu_j + p_j\nu_i}{2}$, $\mathscr{T}^D(p) = \mathscr{C}(p) - \operatorname{tr}(\mathscr{C}(p))\,\mathrm{id}$.

[2] We denote by $\tilde{u}$ the extension of u to Ω_0 by setting $u' = U_0$ in Ω'; $\tilde{u} \in \mathrm{BD}(\Omega_0)$; and $\operatorname{div}\tilde{u} \in L^2(\Omega_0)$ if and only if $u\cdot n = u_0 \cdot \vec{n}_{/\Gamma_0}$.

Therefore we may write Problem (2.33) in the following form:

$$\inf_{\substack{\tilde{u}\in \mathrm{BD}(\Omega_0)\\ \operatorname{div}\tilde{u}\in L^2(\Omega_0)\\ u=u_0|\Omega'}} \left\{ \int_{\Omega\cup\Gamma_0} \phi^D(\varepsilon^D(u)) + \frac{K_0}{2}\int_\Omega (\operatorname{div} u)^2 - \int_{\Omega\cup\Gamma_0} \varepsilon^D(u):\sigma^D \right.$$
$$\left. - \tfrac{1}{3}\int_\Omega (\operatorname{div} u)\,\sigma + \int_{\Gamma_0} \sigma\cdot n u_0 \right\}. \tag{2.41}$$

We now state a theorem which is the analogue of Theorems 2.1.

Theorem 2.4. *Assume that there is a minimizing sequence* $(\tilde{u}_m)$ *for Problem* (2.41) *which is bounded in* $\mathrm{BD}(\Omega_0)$, *and that* $\operatorname{div}(\tilde{u}_m)$ *is bounded in* $L^2(\Omega)$. *Then we may extract from it is a subsequence, still denoted* $(\tilde{u}_m)$, *such that*

$$\tilde{u}_m \to u \quad \text{in } L^1(\Omega_0),$$
$$\phi^D(\varepsilon^D(\tilde{u}_m)) \to \phi^D(\varepsilon^D(u)) \quad \text{vaguely on } \Omega_0,$$
$$\operatorname{div}\tilde{u}_m \to \operatorname{div} u \quad \text{in } L^2_{\mathrm{loc}}(\Omega_0),$$
$$\tilde{u}_m \to u \quad \text{in } L^{N/N-1}_{\mathrm{loc}}(\Omega_0),$$

and u *is a solution of Problem* (2.41).

Proof. The proof follows that of Theorem 2.1, by replacing ψ by ϕ and ∇ by ε. We show, as in Section 1, that if $(\tilde{u}_m)$ is a minimizing sequence for Problem (2.41) which is bounded in $\mathrm{BD}(\Omega)$, and $\operatorname{div}\tilde{u}_m$ is bounded in L^2, then for a subsequence $(\tilde{u}_m)$, $\tilde{u}_m \to u$ in $\mathrm{BD}(\Omega_0)$ weakly, $\operatorname{div}\tilde{u}_m \to \operatorname{div} u$ in $L^2(\Omega_0)$ weakly, and $\phi(\varepsilon(\tilde{u}_m)) \to \mu \geqq 0$ vaguely in $M^1(\Omega_0)$, where μ satisfies $\phi(\varepsilon(u)) \leqq \mu$. Proceeding as with the gradient, we obtain for all τ in $\mathscr{K}\cap\mathscr{S}_{\mathrm{ad}}$

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} ((\phi(\varepsilon(u)) - \varepsilon(u):\tau) &\leqq \int_{\Omega\cup\Gamma_0}\mu - \int_{\Omega\cup\Gamma_0}\varepsilon(u):\tau\\
&\leqq \underline{\lim}\int_{\Omega\cup\Gamma_0}\phi(\varepsilon(u_{m'})) - \int_{\Omega\cup\Gamma_0}\varepsilon(u_{m'}):\tau\\
&= \inf P + \int_{\Gamma_0}\tau\cdot n u_0\\
&\leqq \int_{\Omega\cup\Gamma_0}\phi(\varepsilon(u)) - \int_{\Omega\cup\Gamma_0}\varepsilon(u):\tau.\end{aligned}$$

Then we obtain

$$\begin{aligned}\mu &= \lim \phi(\varepsilon(u_{m'}))\\ &= \phi(\varepsilon(u)), \quad \text{on } \Omega\cup\Gamma_0\end{aligned}$$

and u is a solution for (2.34). This easily implies that $(\operatorname{div} u)^2 = \lim(\operatorname{div} u_m)^2$ vaguely, and $\varepsilon^D(u_m) \to \varepsilon^D(u)$ vaguely. The conclusion follows by Theorem 1.1. □

We now give the analogue of Theorem 2.2.

Theorem 2.5. *Assume that* $F_t \in L^\infty(\Gamma_1)$, $F \cdot n \in L^\infty \cup H^{1/2}(\Gamma)$, *and that there are* $\delta > 0$ *and* σ *with* $\sigma^D \in \frac{\mathscr{K}}{1+\delta}$, $\operatorname{div} \sigma \in L^N(\Omega)$, $\sigma \cdot n = F/\Gamma_1$ [1]. *Let us suppose there is a minimizing sequence* $(\tilde{u}_m)$ *for Problem* (2.33) *which is bounded in* $\mathrm{BD}(\Omega_0)$ *and* $\operatorname{div} \tilde{u}_m$ *is bounded in* $L^2(\Omega_0)$. *Then we may extract from* $(\tilde{u}_m)$ *a subsequence, likewise denoted by* $(\tilde{u}_m)$, *such that*

$$\tilde{u}_m \to u \quad \text{in } L^{N/N-1}(\Omega_0),$$

$$\phi(\varepsilon^D(\tilde{u}_m)) \to \phi(\varepsilon^D(u)) \quad \text{in } M^1(\Omega_0),$$

$$\operatorname{div} \tilde{u}_m \to \operatorname{div} u \text{ in } L^2(\Omega_0) \text{ tightly on } \Omega$$

and

$$u_{m|\Gamma_1} \to u \mid \Gamma_1 \text{ in } L^1(\Gamma_1).$$

Remark 2.2. The Theorem 2.2 improves Theorem 2.1 mainly in that u_m tends to u in $L^1(\Gamma_1)$, because weak convergence in $\mathrm{BD}(\Omega_0)$ does not generally imply convergence of the trace on the boundary.

Let us now give an important consequence of Theorem 2.5. Consider the problem

$$\inf_{\substack{\tilde{u} \in \mathrm{BD}(\Omega_0) \\ \operatorname{div} \tilde{u} \in L^2(\Omega_0) \\ u = u_0 | \Omega'}} \left\{ \int_\Omega \phi(\varepsilon(\tilde{u})) - \lambda \int_\Omega f \cdot u - \lambda \int_{\Gamma_1} F \cdot u \right\} \tag{2.42}$$

and define the real numbers

$$\bar{\bar{\lambda}} = \sup_{\substack{\exists \sigma, \sigma^D \in K^D, \operatorname{div} \sigma \in L^N \\ \sigma \cdot n = F/\Gamma_1}} \{\lambda\}$$

and

$$\underline{\underline{\lambda}} = \sup_{\substack{\exists \sigma, \sigma^D \in K^D, \operatorname{div} \sigma \in L^N \\ \sigma \cdot n = F/\Gamma_1}} \{-\lambda\} .$$

It is obvious that $\bar{\lambda} \leqq \bar{\bar{\lambda}}$, $-\underline{\lambda} \geqq -\underline{\underline{\lambda}}$. It may be easily seen that the strict inequality $\bar{\lambda} < \bar{\bar{\lambda}}$ implies that there are $\delta > 0$ and $\sigma \in L^N\left(\Omega, \frac{K}{1+\delta}\right)$ such that $\operatorname{div} \sigma \in L^N$ and $\sigma \cdot n = \bar{\lambda} F/\Gamma_1$. We then obtain

Corollary 2.3. *Assume that* $0 \leqq \lambda \leqq \bar{\lambda} \leqq \bar{\bar{\lambda}}$ (*respectively* $-\underline{\underline{\lambda}} < -\underline{\lambda} \leqq \lambda \leqq 0$) *and that there is a minimizing sequence* $(\tilde{u}_m)$ *for Problem* (2.42) *that is bounded in*

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[1] If $F \cdot n \in H^{1/2}(\Gamma)$, it suffices to suppose that $(\sigma \cdot n)_t = F_t$.

$\mathrm{BD}(\Omega_0)$. *Then for a subsequence, likewise denoted by* $(\tilde{u}_m)$, *we have*

$$\tilde{u}_m \to u \quad \textit{in } \mathrm{BD}(\Omega_0) \textit{ weakly},$$
$$\phi(\varepsilon^D(\tilde{u}_m)) \to \phi(\varepsilon^D(u)) \quad \textit{tightly on } \Omega_0,$$
$$\operatorname{div} \tilde{u}_m \to \operatorname{div} u \ \textit{ in } L^2(\Omega_0),$$
$$\tilde{u}_{m|\Gamma_1} \to u \mid \Gamma_1$$

and u/Ω *is a solution of Problem* (2.42).

Remark 2.3. When $\lambda \in]-\bar{\lambda}, \bar{\lambda}[$, it is shown in [17] that every minimizing sequence of (2.42) is bounded in $\mathrm{BD}(\Omega_0)$, and as in Remark 2.2 we can give a simpler proof of Corollary (2.3).

Proof of Theorem 2.5. Let $\lambda > 1$ be such that $\tau = \lambda\sigma$ satisfies $\tau^D \in K^D$. Let $(\tilde{u}_m)$ be a minimizing sequence which converges weakly to u in $\mathrm{BV}(\Omega_0)$. As before we have

$$\int_{\Omega \cup \Gamma_0} \phi(\varepsilon(u)) - \int_{\Omega \cup \Gamma_0} \varepsilon(u) : \tau \leqq \underline{\lim} \int_{\Omega \cup \Gamma_0} \phi(\varepsilon(\tilde{u}_m)) - \int_{\Omega \cup \Gamma_0} \varepsilon(\tilde{u}_m) : \tau .$$

Thus if σ is the solution of the dual problem (2.37), we get by Theorem 2.5

$$\phi(\varepsilon(\tilde{u}_m)) - \varepsilon(\tilde{u}_m) : \sigma \to -\phi^*(\sigma) = \phi(\varepsilon(u)) - \varepsilon(u) : \sigma \qquad \text{on } \Omega \cup \Gamma_0 .$$

Therefore

$$\int_{\Omega \cup \Gamma_0} (\phi(\varepsilon(u)) - \varepsilon(u) : \tau)$$
$$\leqq \underline{\lim} \int_{\Omega \cup \Gamma_0} \varepsilon(\tilde{u}_m) : (\sigma - \tau) + \int_{\Omega \cup \Gamma_0} (\phi(\varepsilon(\tilde{u}_m)) - \varepsilon(u_m) : \sigma);$$

which implies

$$\int_{\Gamma_1} Fu \leqq \underline{\lim} \int_{\Gamma_1} Fu_m .$$

The reversed inequality is satisfied without using the assumption on σ, so we get $\int_{\Gamma_1} Fu = \lim_m \int_{\Gamma_1} Fu_m$, and hence $\lim \int_{\Omega \cup \Gamma_0} \phi(\varepsilon(\tilde{u}_m)) = \int_{\Omega \cup \Gamma_0} \phi(\varepsilon(\tilde{u}))$. By applying Theorem 1.2 in Part I, we conclude, in addition, that $u_{m|\Gamma_1} \to u \mid \Gamma_1$. $\square$

Let us now see how the conclusions in Part I may be applied to limit analysis in plasticity.

We begin by recalling the relaxed form of the problem for perfect plasticity:

$$\inf_{\substack{v \in \mathrm{BD}(\Omega) \\ Lv = 1 \\ \operatorname{div} v = 0, v \cdot n = 0/\Gamma_0}} \left\{ \int_{\Omega} \phi_\infty(\varepsilon^D(v)) + \int_{\Gamma_0} \phi_\infty(\mathscr{C}^D(-v)) \right\}, \tag{2.43}$$

which may also be written in the form

$$\inf_{\substack{Lv = 1 \\ \tilde{v} \in \mathrm{BD}(\Omega_0) \\ \tilde{v} = 0/\Omega' \\ \operatorname{div} v = 0}} \left\{ \int_{\Omega_1 \cup \Gamma_0} \phi_\infty(\varepsilon^D(\tilde{v})) \right\}. \tag{2.44}$$

The dual problem of (2.37) is

$$\sup_{\substack{\exists\sigma,\sigma^D\in K^D\\ \operatorname{div}\sigma+\lambda f=0/\Omega\\ \sigma\cdot n=F/\Gamma_1}}\{\lambda\} = \bar{\lambda}. \tag{2.44}$$

We recall that $\bar{\bar{\lambda}}$ is defined as

$$\bar{\bar{\lambda}} = \sup_{\substack{\exists\sigma,\sigma^D\in K^D\\ \operatorname{div}\sigma\in L^N\\ (\sigma\cdot n)=\lambda F}}\{\lambda\}.$$

The expected result is the following

Theorem 2.6. *Let us suppose that* $\bar{\lambda} < \bar{\bar{\lambda}}$. *Then Problem* (2.43) *has a solution* u *in* $\mathrm{BD}(\Omega)$ *with* $\operatorname{div} u \in L^2(\Omega)$. *More precisely, if* $(\tilde{u}_m)$ *is a minimizing sequence for Problem* (2.43) *we may extract from it a subsequence* $(\tilde{u}_m)$ *such that*

$$\tilde{u}_m \to u \quad \text{in } \mathrm{BD}(\Omega_0) \textit{ weakly},$$

$$\phi^D_\infty(\varepsilon^D(\tilde{u}_m)) \to \phi^D_\infty(\varepsilon^D(u)) \quad \textit{tightly on } \Omega_0$$

and consequently $\tilde{u}_m \to u$ *in* $L^1(\Gamma_1)$, $Lu = 1$ *and* $\tilde{u}_m \to u$ *in* $L^{N/N-1}(\Omega_0)$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 2.5. If $(\tilde{u}_m)$ is a minimizing sequence for Problem (2.42), $\varepsilon^D(\tilde{u}_m)$ is bounded in $M^1(\Omega)$, and since $\operatorname{div}\tilde{u}_m = 0$, we may extract from $\tilde{u}_m$ a subsequence, also denoted by $(\tilde{u}_m)$, such that

$$\tilde{u}_m \to u \quad \text{in } \mathrm{BD}(\Omega_0) \text{ weakly}.$$

We have $\operatorname{div} u = 0$ and if $\bar{\sigma}$ is a solution of the dual problem we obtain, as in the proof of Theorem 2.5,

$$\phi^D_\infty(\varepsilon^D(\tilde{u})) = \varepsilon^D(\tilde{u}) : \bar{\sigma} \quad \text{on } \Omega \cup \Gamma_0.$$

Then if $u \neq 0$, u/Lu is a solution for Problem (2.43). We may in fact show, as in the proof of Theorem 2.3, that $Lu_m \to Lu$ and finally obtain the expected conclusion. □

2.3. Application to the theory of elastic-perfectly plastic plates

Let us consider a plate which occupies a bounded domain Ω of $\mathbb{R}^2$ ($\partial\Omega$ is supposed to be C^2). We assume for simplicity that this plate is subjected to null force and moment on the boundary, and to a surface load of the form

$$f = g + \Sigma\, \mu_i\, \delta_i$$

where $\mu_i \in l^1(\mathbb{R})$[1], δ_i is the Dirac measure on (x_i) and $g \in L^2(\Omega)$, the x_i's having only a finite number of cluster points on every compact subset of Ω. In consequence of the study made in [6], the variational problem for elastic-perfectly plastic plates may be written in the form

$$\inf_{u \in \mathrm{HB}(\Omega)} \left\{ \int_\Omega \psi(\nabla\nabla u) - \int_\Omega fu \right\} \tag{2.45}$$

where ψ is the conjugate of ψ^*:

$$\psi^* = \begin{cases} \frac{1}{2} AM : M & \text{if } M \in K \\ +\infty & \text{if } M \notin K, \end{cases}$$

K is a bounded convex set of K which contains O in its interior and A is a positive definite operator on E.

It is shown in [6] that the dual problem of (2.45) is simply

$$\sup_{\substack{\nabla\cdot\nabla\cdot M + g = 0 \text{ in } \Omega \\ b_1(M) = 0/\Gamma, M \in K \\ b_0(M) = 0/\Gamma}} \left\{ -\int_\Omega \psi^*(M) \right\}. \tag{2.46}$$

Moreover, when $M \in L^\infty(\Omega, E)$, $\nabla \cdot \nabla \cdot M(\Omega)$ and u belongs to $\mathrm{HB}(\Omega)$, we defined in [6] the measure $\nabla\nabla u : M$ on Ω and proved

Proposition 2.3. i) *Let $\nabla\nabla u : M$ be the distribution defined as*

$$\int_\Omega \nabla\nabla u : M\varphi = \int_\Omega u \nabla \cdot \nabla \cdot M\varphi - 2 \int_\Omega (\nabla u \otimes \nabla\varphi) : M - \int_\Omega u(\nabla\nabla\varphi) : M,$$

for every φ in $\mathscr{D}(\Omega)$. Then $\nabla\nabla u : M$ may be extended as a bounded measure on Ω, such that

$$|\nabla\nabla u : M| \leqq |\nabla\nabla u| \, |M|_\infty .$$

ii) *Green's formula holds in the following form:*

$$\int_\Omega (\nabla\nabla u : M)\, \varphi = \int_\Omega u \nabla \cdot \nabla \cdot M\varphi + \int_\Gamma b_1(M) \left[\frac{\partial u}{\partial n} \varphi + u \frac{\partial \varphi}{\partial n} \right] - \int_\Gamma b_0(M)\, u\varphi,$$

for every φ in $\mathscr{C}^1(\bar{\Omega})$.

We now establish the existence of a minimizer for Problem 2.45.

Theorem 2.7. *Assume that $\langle f, p \rangle = 0$ for all p in $\mathscr{P}_1$*[2]*, and suppose that there is a minimizing sequence (u_m) for Problem (2.45) which is bounded in $\mathrm{HB}(\Omega)/\mathscr{P}_1$. Then there is a subsequence, likewise denoted by (u_m), and a sequence (p_m) in $\mathscr{P}_1$, such that*

$$u_m - p_m \to u \quad \textit{in } \mathrm{HB}(\Omega) \textit{ weakly},$$

$$\psi(\nabla\nabla u_m) \to \psi(\nabla\nabla u) \quad \textit{vaguely on } \Omega \setminus \cup \{x_i\}.$$

[1] $l^1(\mathbb{R})$ denotes the class of absolutely convergent series.

[2] $\mathscr{P}_1$ denotes the space of the affine functions on Ω, or, equivalently, $\mathscr{P}_1 = \{T \in \mathscr{D}'(\Omega, E)$ $\nabla\nabla T = 0\}$.

Moreover $u_m - p_m$ *is pointwise convergent to* u *on* $\Omega \setminus \bigcup_i \{x_i\}$ *and* u *is a solution for Problem* (2.40).

Proof. Let us note first that by virtue of the generalized Green's formula in Proposition 2.3, we may write

$$\int_\Omega \nabla\nabla u : M = \int_\Omega fu + \int_\Gamma b_1(M)\frac{\partial u}{\partial n} - \int_\Gamma b_0(M)\, u = \int_\Omega fu,$$

for M in $\mathscr{S}_{\mathrm{ad}}$ and u in $\mathrm{HB}(\Omega)$. Therefore, Problem (2.45) may be written in the form

$$\int_\Omega \psi(\nabla\nabla u) - \int_\Omega \nabla\nabla u : M.$$

Now let (u_m) be a minimizing sequence for Problem (2.45), which is bounded in $\mathrm{HB}(\Omega)$. We may extract from it a subsequence, again denoted by (u_m), and take $p_m \in \mathscr{P}$ such that

$$u_m - p_m \to u \quad \text{in } W^{1,1}(\Omega) \text{ (strongly)},$$
$$\nabla\nabla u_m \to \nabla\nabla u \quad \text{in } M^1(\Omega, E) \text{ vaguely},$$

(*cf.* [7], [6]). We fix $\varepsilon > 0$. Then we select a positive number δ and we denote by Ω_δ the open set

$$\Omega_\delta = \{x \in \mathbb{R}^N, d(x, \Gamma) < \delta\}.$$

We construct φ_δ in $\mathscr{D}(\Omega_\delta)$, $\varphi_\delta = 1$ on Γ, $0 \leqq \varphi_\delta \leqq 1$, and we set $g_\delta = 1 - \varphi_\delta$. We choose $\delta > 0$ so small that

$$\int_{\Omega_\delta} |\nabla\nabla \tilde{u}| < \varepsilon, \tag{2.47}$$

$$\int_{\Omega_\delta \cap \Omega} \psi^*(M) < \varepsilon \tag{2.48}$$

where $M \in L^\infty(\Omega, K)$, $M \in \mathscr{S}_{\mathrm{ad}}$. We denote by J_δ the finite subset of I, $J_\delta = \{j \in I \mid x_j \text{ is a cluster point in } \Omega \setminus \Omega_\delta\}$. Since $\int_{\{x_j\}} |\nabla\nabla u| = 0 \;\; \forall j \in I$, we may choose $\delta_1 > 0$ so that

$$\int_{\bigcup_{j\in J_\delta} B(x_j,\delta_1)} |\nabla\nabla u| < \varepsilon \quad \text{and} \quad \int_{\bigcup_{j\in J_\delta} B(x_j,\delta_1)} \psi^*(M) < \varepsilon. \tag{2.49}$$

On the other hand there is an $N \in \mathbb{N}$ such that

$$x_n \in \bigcup_{j\in J} B(x_j, \delta_{1/2}) \cup \Omega_\delta, \quad n > N \tag{2.50}$$

and we may choose δ_2 such that

$$\int_{\bigcup_{j=1}^N B(x_j,\delta_2)} |\nabla\nabla u| < \varepsilon, \quad \text{and} \quad \int_{\bigcup_{j=1}^N B(x_j,\delta_2)} \psi^*(M) < \varepsilon. \tag{2.51}$$

Let now h_δ be $\mathscr{C}^\infty$ with compact support in $\bigcup_{j\in J_\delta} B(x_j, \delta_1) \bigcup_{j\in[1,N]} B(x_j, \delta_2)$, $h_\delta = 1$ on $\left(\bigcup_{j\in J_\delta} B(x_j, \delta_{1/2})\right) \cup \left(\bigcup_{j\in[1,N]} B(x_j, \delta_{2/2})\right)$, $0 \leqq h_\delta \leqq 1$. We may write

$$\int_\Omega \psi(\nabla\nabla u) - \int_\Omega \nabla\nabla u : M = \int_\Omega (\psi(\nabla\nabla u) - (\nabla\nabla u : M))\,(\varphi_\delta + g_\delta h_\delta)$$
$$+ \int_\Omega (\psi(\nabla\nabla u) - (\nabla\nabla u : M))\, g_\delta(1 - h_\delta).$$

By lower semi-continuity,

$$\int_\Omega \psi(\nabla\nabla u)\, g_\delta(1 - h_\delta) \leqq \underline{\lim} \int_\Omega \psi(\nabla\nabla u_m)\, g_\delta(1 - h_\delta),$$

and since $1 - h_\delta = 0$ on every $x_i \in \operatorname{Supp}^t g_\delta$, Green's formula in Proposition 2.3 implies that

$$\int_\Omega (\nabla\nabla \tilde{u}_m : M)\, g_\delta(1 - h_\delta) \to \int_\Omega (\nabla\nabla \tilde{u} : M)\, g_\delta(1 - h_\delta).$$

We now focus our attention on the term $(\psi(\nabla\nabla u) - (\nabla\nabla u : M)\,(g_\delta h_\delta + \varphi_\delta)$. Using (2.47), (2.49) and (2.51) we have

$$\int_\Omega (\psi(\nabla\nabla u) - (\nabla\nabla u : M))\,(\varphi_\delta + g_\delta h_\delta)$$
$$\leqq \int_{\operatorname{supp}^t\varphi_\delta \cup \operatorname{supp}^t h_\delta} \psi(\nabla\nabla u) + |M|_\infty \int_{\operatorname{supp}^t\varphi_\delta \cup \operatorname{supp}^t h_\delta} |\nabla\nabla u|$$
$$\leqq c_1 \int_{\operatorname{supp}^t\varphi_\delta \cup \operatorname{supp}^t h_\delta} |\nabla\nabla u|$$
$$\leqq c_1 \left[\int_{\Omega_\delta} |\nabla\nabla u| + \int_{\bigcup_{j\in J_\delta} B(x_j,\delta_1)} |\nabla\nabla u| + \int_{\bigcup_{j\in J_\delta}^{N} B(x_j,\delta_2)} |\nabla\nabla u| \right]$$
$$\leqq 3c_1\varepsilon.$$

By recalling the previous inequalities, we get

$$\int_\Omega (\psi(\nabla\nabla u) - (\nabla\nabla u : M)) \leqq 3c_1\varepsilon + \int_\Omega (\psi(\nabla\nabla u) - (\nabla\nabla u : M))\, g(1 - h_\delta)$$
$$\leqq 3c_1\varepsilon + \underline{\lim} \int_\Omega (\psi(\nabla\nabla u_m) - (\nabla\nabla u_m : M))\, g_\delta(1 - h_\delta)$$
$$\leqq 3c_1\varepsilon + \underline{\lim} \int_\Omega (\psi(\nabla\nabla u_m) - (\nabla\nabla u_m : M))$$
$$+ \int_\Omega \psi^*(M)\,(\varphi_\delta + g_\delta h_\delta)$$
$$\leqq \underline{\lim} \int_\Omega (\psi(\nabla\nabla u_m) - (\nabla\nabla u_m : M)) + 3c_1\varepsilon$$
$$+ \int_{\Omega_\delta \left[\bigcup_{j\in J_\delta} B(x_j,\delta_1) \bigcup_{j=1}^{N} B(x_j,\delta_2)\right]} (\psi^*(M))$$

$$\leqq \underline{\lim} \int_\Omega \psi(\nabla\nabla u_m) - (\nabla\nabla u_m : M) + (3c_1 + 1)\,\varepsilon$$

(by (2.48), (2.49) and (2.51))

$$= \inf P + (3c_1 + 1)\,\varepsilon.$$

Since ε is arbitrary, we deduce that u is a solution of Problem (2.45). The inequality $\psi(\nabla\nabla u) - \nabla\nabla u : M \geqq -\psi^*(M)$, which holds for any $M \in K \cap \mathscr{S}_{\mathrm{ad}}$, becomes an equality when $M = \overline{M}$, namely a solution for the dual problem (2.46).

Since $\int_\Omega (\psi(\nabla\nabla u_m) - \nabla\nabla u_m : \overline{M})$ converges to $-\int_\Omega \psi^*(\overline{M})$, by using the inequality $\psi(\nabla\nabla u_m) - \nabla\nabla u_m : \overline{M} \geqq -\psi^*(\overline{M})$ on Ω we obtain

$$\psi(\nabla\nabla u_m) - \nabla\nabla u_m : \overline{M} \to -\psi^*(\overline{M}) = \psi(\nabla\nabla u) - \nabla\nabla u : \overline{M} \quad \text{(tightly on } \Omega).$$

Since $\nabla\nabla u_m : \overline{M} \to \nabla\nabla u : \overline{M}$ vaguely on $\Omega \setminus \bigcup_i \{x_i\}$, we conclude, in addition, that $\psi(\nabla\nabla u_m) \to \psi(\nabla\nabla u)$ vaguely on $\Omega \setminus \bigcup_i \{x_i\}$. Theorem 1.3 then asserts that (u_m) is pointwise convergent to u on $\Omega \setminus \bigcup_i \{x_i\}$.

We now give the analogues of Theorems 2.2 and 2.5. We begin by defining the real numbers $\bar{\lambda}$ and $\bar{\bar{\lambda}}$ by

$$\bar{\lambda} = \sup_{\substack{\exists M \in L^\infty(\Omega, K) \\ \nabla\cdot\nabla\cdot M = \lambda\left(g + \sum_i \mu_i \delta_i\right) \\ b_1(M) = 0/\Gamma,\ b_0(M) = 0/\Gamma}} \{\lambda\}, \tag{2.52}$$

$$\bar{\bar{\lambda}} = \sup_{\substack{\exists M \in L^\infty(\Omega, K) \\ \nabla\cdot\nabla\cdot M - \lambda \sum_i \mu_i \delta_i \in L^1(\Omega) \\ b_1(M) = 0/\Gamma,\ b_0(M) = 0/\Gamma}} \{\lambda\} \tag{2.53}$$

and consider the problem

$$\inf_{u \in \mathrm{HB}(\Omega)} \left(\int_\Omega \psi(\nabla\nabla u) - \lambda \left[\int_\Omega g u + \sum_i \mu_i \delta_i \right] \right). \tag{2.54}$$

Theorem 2.8. *Assume that* $\lambda < \bar{\bar{\lambda}}$ *and that there is a minimizing sequence* (u_m) *for* (2.54) *which is bounded in* HB. *Then we may extract from it a subsequence such that*

$$u_m \to u \quad \textit{in HB weakly}$$

and

$$u_m \quad \textit{is pointwise convergent to } u \textit{ in } \Omega.$$

Proof. Taking $\lambda' \in [\lambda, \bar{\bar{\lambda}}[$, and M in $L^\infty(\Omega, K)$ such that $\nabla \cdot \nabla \cdot M - \lambda' \Sigma \mu_i \delta_i \in L^1(\Omega)$, we have, following the proof of Theorem 2.1,

$$\int_\Omega (\psi(\nabla\nabla u) - \nabla\nabla u : M) \leqq \underline{\lim} \int_\Omega (\psi(\nabla\nabla u_m) - (\nabla\nabla u_m : M))$$

$$= \underline{\lim} \int_\Omega (\psi(\nabla\nabla u_m) - (\nabla\nabla u_m : \bar{M})) + \underline{\lim} \int_\Omega (\nabla\nabla u_m : (\bar{M} - M))$$

$$= \int_\Omega (\psi(\nabla\nabla u) - (\nabla\nabla u : \bar{M})) + \underline{\lim} \int_\Omega \nabla\nabla u_m : (\bar{M} - M).$$

This implies that $\overline{\lim} \int_\Omega \nabla\nabla(u - u_m) : (\bar{M} - M) \leqq 0$. The reverse inequality is true without using $\lambda < \bar{\bar{\lambda}}$ and may be obtained as follows:

$$\int_\Omega \nabla\nabla u : \bar{M} = \int_\Omega \psi(\nabla\nabla u) + \int_\Omega \psi^*(\bar{M})$$

$$\leqq \underline{\lim} \int_\Omega (\psi(\nabla\nabla u_m) + \psi^*(\bar{M}))$$

$$= \underline{\lim} \int_\Omega (\nabla\nabla u_m : \bar{M}).$$

Consequently, $\overline{\lim} \int \nabla\nabla(u - u_m) : \bar{M} \leqq 0$, so $\lim \int \nabla\nabla(u - u_m) : \bar{M} = 0$. By using Green's formula and $\bar{M} \in \mathscr{S}_{\text{ad}}$, we get

$$\lim_m \sum_i \mu_i (u - u_m)(x_i) = 0.$$

We have thus obtained

$$\lim \int_\Omega \psi(\nabla\nabla u_m) = \lim \int_\Omega \nabla\nabla u_m : \bar{M} + \int_\Omega \psi(\nabla\nabla u) - \int_\Omega \nabla\nabla u : \bar{M} = \int_\Omega \psi(\nabla\nabla u),$$

which implies that $(\psi(\nabla\nabla u_m))$ converges tightly to $\psi(\nabla\nabla u)$ and that (u_m) is pointwise convergent to u in Ω (by Corollary 1.1). $\square$

Remark 2.5. By Proposition 1.2 we conclude that $\gamma_0(u_m)$ converges to $\gamma_0(u)$ in $\gamma_0(W^{2,1})$, and that $\gamma_1(u_m)$ converges to $\gamma_1(u)$ in $L^1(\Gamma)$.

Remark 2.6. We could have treated the more general case where the plate is subjected to boundary conditions

$$u = u_{0/\Gamma_0 \cup \Gamma_1}$$

$$\frac{\partial u}{\partial n} = u_{1/\Gamma_0}$$

$$b_1(M) = D^1/\Gamma_2 \cup \Gamma_1$$

$$b_0(M) = D^0/\Gamma_2,$$

with f being as before. The proof is then a combination of the arguments used in the proof of Theorem (4.2) in [6] and of Theorem 2.8 here (Sections 2.1 and 2.2).

We now consider the limit analysis problem for plates. We begin by recalling the relaxed limit analysis problem for plates:

$$\inf_{\substack{\langle f,u\rangle=1 \\ u\in \mathrm{HB} \text{ or } \mathrm{HB}/\mathscr{P}^1}} \left\{ \int_\Omega \psi_\infty(\nabla\nabla u) \right\}. \tag{2.55}$$

The study made in [6] showed that

$$\inf(2.55) = \bar{\lambda}.$$

We now establish existence of a solution.

Theorem 2.9. *Assume that* $\bar{\lambda} < \bar{\bar{\lambda}}$. *Then a solution u of Problem* (2.55) *exists. Moreover every minimizing sequence converges pointwise to u.*

Proof of Theorem 2.9. Let us take a minimizing sequence (u_m) for Problem (2.55); (u_m) is bounded in $\mathrm{HB}/_{P_1}$ so there are a (p_m) in P_1^{N} and a subsequence, also denoted by (u_m), such that

$$u_m - p_m \to u \quad \text{in HB weakly}.$$

We may take $Lu_m = 1$. Let us now consider M such that $\lambda M \in K$, for some $\lambda > 1$, and $\nabla \cdot \nabla \cdot M - \Sigma \mu_i \delta_i \in L^1$. We have

$$\begin{aligned} \int_\Omega \psi_\infty(\nabla\nabla u) - \int_\Omega \nabla\nabla u : \lambda M &\leq \underline{\lim} \int_\Omega \psi_\infty(\nabla\nabla u_m) - \int_\Omega \nabla\nabla u_m : \lambda M \\ &= \lim \int_\Omega \psi_\infty(\nabla\nabla u_m) - \int_\Omega \nabla\nabla u_m : \bar{M} + \int_\Omega \nabla\nabla u_m : (M - \lambda\bar{M}) \\ &= \int_\Omega \psi_\infty(\nabla\nabla u) - \int_\Omega \nabla\nabla u : \bar{M} + \underline{\lim} \int_\Omega \nabla\nabla u_m : (M - \lambda\bar{M}), \end{aligned}$$

where $\bar{M}$ is a solution for the dual problem (2.52). Following the proof of Theorem 2.8, we get $\psi_\infty(\nabla\nabla u) = \nabla\nabla u : \bar{M}$ and then

$$\int \nabla\nabla u : (M - \lambda M) \leq \underline{\lim} \int \nabla\nabla u_m : (\bar{M} - \lambda M).$$

The opposite inequality holds without the assumption $\bar{\lambda} < \bar{\bar{\lambda}}$ on $\bar{\lambda}$, as may be seen by

$$\int_\Omega \nabla\nabla u : M = \int_\Omega \psi(\nabla\nabla u) \leq \lim \int_\Omega \psi(\nabla\nabla u_m) = \underline{\lim} \int_\Omega \nabla\nabla u_m : M.$$

By Green's formula in Proposition 2.3, these inequalities imply that

$$\sum_i \mu_i u(x_i) = \underline{\lim} \sum_i \mu_i u_m(x_i).$$

We thus obtain $Lu = \lim Lu_m = 1$ and that $\int_\Omega \psi(\nabla\nabla u_m) \to \int_\Omega \psi(\nabla\nabla u)$. This implies, by virtue of Corollary 1.1, that $u_m(x) \to u(x)$, for every x in Ω.

We conclude with a brief discussion of the case $\bar{\lambda} = \bar{\bar{\lambda}}$. For more details, the reader may consult my thesis [26]. In [19] SUQUET proposed another form of a relaxed problem in limit analysis. For definiteness, we state it here for the case of antiplane shear in plasticity:

$$\inf_{\substack{u \in \mathrm{BV}(\Omega) \\ u^+ \in L^1(\Gamma_1) \\ \int_\Omega fu + \int_{\Gamma_1} Fu^+ = 1}} \left\{ \int_\Omega \psi_\infty(\nabla u) + \int_{\Gamma_0} \phi_\infty(-u\vec{n}) + \bar{\lambda} \int |F| \, |u^+ - u^-| \right\}. \tag{2.56}$$

This is a convenient relaxed form for Problem (2.28), since, as we showed in [26], Chapter III, $\inf(2.28) = \inf(2.56)$ and (2.56) admits a solution (u, u^+). By contrast, as shown in [26] by means of a counterexample, Problem (2.28) does not generally have a solution when $\bar{\lambda} = \bar{\bar{\lambda}}$. Furthermore, when (2.28) does not have a minimizer, the only solutions of (2.56) are of the form $(0, \alpha\vec{F})$ where $\alpha \in L^1(\Gamma_1)$ and $\int \alpha F^2 = 1$. The proof of these results and their analogues in plasticity and plate theory are given in [26].

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