Some Existence's results for non coercive "1-Laplacian" operator.

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Abstract

In this paper we study necessary and sufficient conditions on f and the first eigenfunctions for the 1-Laplacian, for equations of the form

$$\begin{cases} -\mathrm{div}(\sigma) - \lambda = f u^{q-1} & u \geq 0 \text{ in } \Omega, \\ \sigma \cdot \nabla u = |\nabla u|, \; |\sigma|_{L^\infty(\Omega)} \leq 1, \; u \in W_0^{1,1}(\Omega) \end{cases}$$

when $q \leq \frac{N}{N-1}$ and $\lambda \geq \lambda_1$, λ_1 is the first eigenvalue for the 1-Laplacian.

1 Introduction

In this paper we are interested in the existence of solutions for the following partial differential equation

$$\begin{cases} -\mathrm{div}(\sigma) - \lambda = f u^{q-1}, & u \geq 0 \text{ in } \Omega, \\ \sigma \cdot \nabla u = |\nabla u|, & |\sigma|_{L^\infty(\Omega)} \leq 1, \; u \in W_0^{1,1}(\Omega) \end{cases} \tag{1}$$

where Ω is some bounded domain in $\mathbb{R}^N$, which is piecewise $\mathcal{C}^1$ and $f \in \mathcal{C}(\Omega) \cap L^\infty(\Omega)$.

The results presented here extend to the case where $p = 1$ the one obtained in [12] for the case $p > 1$.

Previous results about this type of p.d.e. have been developped in [13]. They concern the case where the functional J_λ defined by

$$J_\lambda(u) = \int_\Omega (|\nabla u| - \lambda |u|)$$

is coercive. This means, with the notations here employed, that $\lambda < \lambda_1$, where

$$\lambda_1 = \inf_{u \in W_0^{1,1}(\Omega), \int_\Omega |u|=1} \{ \int_\Omega |\nabla u| \} \tag{2}$$

The properties of λ_1, called first eigenvalue for the 1-Laplacian and of the "minimizers" of (2) are developped in section 2.

The difficulties here are of several types

1) The solutions of (1) are obtained as minima of functionals which are non coercive. This difficulty can be overcome by using a process borrowed to Ouyang [29] in the case of the usual Laplacian and to Birindelli Demengel in [12] for the p-Laplacian.

2) As in the case of coercive problems in $BV(\Omega)$, the lack of compactness of the trace map does not permit to ensure that the boundary condition $u = 0$ holds. This leads to introduce a relaxed version of the variational functional that we minimize, for which one is able to prove the existence of a minimizer. This relaxed term is responsible of the weakened version of the homogeneous boundary conditions "$\sigma.\vec{n}u = -|u|$ on $\partial\Omega$" which appears in equation (5) later.

3) The solutions that we obtain are minima of nondifferentiable functionals, hence difficulty occurs when one tries to exhibit a partial differential equation satisfied by u. The usual process used to find the p.d.e. associated to BV functionals consists in using duality theory. Unfortunately, this cannot be employed here because the constraint $\int_\Omega f|u|^q = 1$ is not a convex constraint. In order to overcome this, one can use an approximation by means of a functional defined and coercive on $W_0^{1,1+\epsilon}(\Omega)$, $\epsilon > 0$. The following steps consists in letting ϵ go to zero.

4) In the critical case, additional difficulties occur when passing to the limit : The existence's result can finally be achieved by using an adaptation of the famous concentration compactness principle of P.L. Lions [26].

Let us make a few remarks and precisions about the setting : First, we need to extend the definition of the p.d.e. (1) in order that it makes sense for functions in $BV(\Omega)$.

Proposition 1 *Suppose that Ω is some bounded domain in $\mathbb{R}^N$ whose boundary is piecewise $\mathcal{C}^1$. Suppose that $u \in BV(\Omega)$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^N)$, is such that div $\sigma \in L^N(\Omega)$. We define the distribution $\sigma \cdot \nabla u$ by the following*
1) For every $\varphi \in \mathcal{D}(\Omega)$

$$\langle \sigma \cdot \nabla u, \varphi \rangle = -\int_\Omega \text{div}(\sigma) u\varphi - \int_\Omega \sigma \cdot \nabla(\varphi) u \tag{3}$$

The distribution $\sigma.\nabla u$ hence defined is a bounded measure in Ω, absolutely continuous with respect to $|\nabla u|$, with

$$|\sigma \cdot \nabla u| \leq |\nabla u||\sigma|_\infty$$

2) The following generalized Green's formula holds for $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\langle \sigma \cdot \nabla u, \varphi \rangle = -\int_\Omega \mathrm{div}(\sigma) u\varphi - \int_\Omega (\sigma \cdot \nabla \varphi) u + \int_{\partial\Omega} \sigma \cdot \vec{n} u \varphi \tag{4}$$

where $\vec{n}$ denotes the unit outer normal to $\partial\Omega$.
3) Define

$$\sigma \cdot (\nabla u)^s = \sigma.\nabla u - \sigma \cdot \nabla u^{ac},$$

where ∇u^{ac} and ∇u^s denote the absolutely continuous and singular part of ∇u. Then, $\sigma.(\nabla u)^s$ is singular and

$$|\sigma \cdot (\nabla u)^s| \leq |(\nabla u)^s||\sigma|_\infty$$

The proof of Proposition 1 can be founded in [34], [13].

We shall say that $u \geq 0$, $u \in BV(\Omega)$ satisfies (1), if

$$\begin{cases} -\mathrm{div}\sigma - \lambda = f u^{q-1} \quad u \geq 0 & \text{in } \Omega, \\ \sigma \cdot \nabla u = |\nabla u|, & \text{in } \Omega \\ \sigma \cdot \vec{n}(-u) = u & \text{on } \partial\Omega \end{cases} \tag{5}$$

We shall denote in the sequel the equation (5) as eq_λ.

Remark 1 *Suppose that $u \in BV(\Omega)$ and define $\tilde{u}$ by*

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^N \backslash \overline{\Omega} \end{cases}$$

Then, $\tilde{u} \in BV(\mathbb{R}^N)$, with

$$\nabla \tilde{u} = \nabla u \chi_\Omega + (-u\vec{n})\delta_{\partial\Omega}$$

with $\delta_{\partial\Omega}$ the uniform Dirac measure on $\partial\Omega$. and

$$|\nabla \tilde{u}| = |\nabla u|\chi_\Omega + |u|\delta_{\partial\Omega}.$$

Finally, let us define the measure $\sigma.\nabla\tilde{u}$ *on* $\Omega\cup\partial\Omega$ *as*

$$\sigma.\nabla\tilde{u} = \sigma.\nabla u\chi_\Omega + \sigma.\vec{n}(-u)\delta_{\partial\Omega}$$

It satisfies

$$\sigma\cdot\nabla\tilde{u} = |\nabla\tilde{u}| \text{ on } \Omega\cup\partial\Omega$$

iff

$$\left\{\begin{array}{l}\sigma\cdot\nabla u = |\nabla u| \text{ in } \Omega,\ |\sigma|_{L^\infty(\Omega)}\leq 1,\\ \sigma\cdot\vec{n}(-u) = u \text{ on } \partial\Omega\end{array}\right. \tag{6}$$

In the sequel we shall drop the tilde on u and say that $\sigma.\nabla u = |\nabla u|$ on $\Omega\cup\partial\Omega$ as soon as (6) is satified.

Remark 2 *Let us observe that except if* $\lambda=\lambda_1$*, we do not need to precise that we look for nontrivial solutions. Indeed,* 0 *is a solution if and only if there exists some* σ*,* $|\sigma|_\infty\leq 1$ *and* $-\mathrm{div}(\sigma)-\lambda=0$*. This cannot happen for* $\lambda>\lambda_1$ *since one has, multiplying by any* $v\in BV(\Omega)$*,* $v\geq 0$*,* $v\neq 0$

$$\lambda\int_\Omega v = \int_\Omega \sigma\cdot\nabla v - \int_{\partial\Omega}\sigma.\vec{n}v \leq \int_{\Omega\cup\partial\Omega}|\nabla v|$$

since $|\sigma|_\infty\leq 1$*, using Proposition 1. Dividing by* $\int_\Omega v$ *one gets that* $\lambda\leq\lambda_1$*.*

The solutions of such equations are particular case of what we called "almost weakly 1-harmonic functions", defined in [15] :

Definition 1 *Let* Ω *be a bounded domain in* $\mathbb{R}^N$*, piecewise* $\mathcal{C}^1$*.* $u\in BV(\Omega)$ *is said to be weakly almost* **1***- harmonic if there exists* $\sigma\in L^\infty(\Omega)$*,* $|\sigma|_\infty\leq 1$*,* $\mathrm{div}(\sigma)\in L^N(\Omega)$ *and*

$$\sigma.\nabla u = |\nabla u|$$

on $\Omega\cup\partial\Omega$*.*

The main properties of such functions which are required in this paper are enumerated in Section 2.

We now present the results here enclosed. First, we prove some necessary conditions on the solution when it exists, and the consequences on f.

Let us define the sets Ω^+ and Ω^- by

$$\Omega^+ = \{x\in\Omega,\ f(x)>0\}$$

$$\Omega^- = \{x\in\Omega,\ f(x)<0\}.$$

Let us denote by $K(N,1)$ the best constant for the Sobolev embedding from $W^{1,1}(\mathbb{R}^N)$ in $L^{1^\star}(\mathbb{R}^N)$. This constant has been computed by Aubin [7], Talenti [36] and has value $K(N,1) = |S_N|^{\frac{-1}{N}} N^{-1+\frac{1}{N}}$.

Theorem 1 *Suppose that there exists a (nontrivial) solution to* eq_λ*, then*

- *1 For* $\lambda > \lambda_1$*,* $\Omega^- \neq \emptyset$*, and for all first eigenfunction* ϕ*,* $\int_\Omega f|\phi|^q < 0$*.*
- *2. For* $\lambda = \lambda_1$*,* $\Omega^+ \neq \emptyset$*.*

Remark 3 *In the case where* $N = 2$ *and* Ω *is a regular convex domain, it is known that there is uniqueness of the first eigenfunction (up to a constant). This is a consequence of the uniqueness of the Cheeger set under these conditions. [7], [10], [3],[4], [30], [32]*

The second result concerns the nonexistence for λ large and the fact that the set of λ for which there exists a solution is an intervall.

Theorem 2 *Suppose that* $N \geq 1$*, and that* $q \leq 1^\star = \frac{N}{N-1}$*. Then, there exists* $\lambda^\star$ *such that for* $\lambda > \lambda^\star$ *no solution exists for* eq_λ*. Moreover, the set of* $\lambda > \lambda_1$ *for which* eq_λ *has a solution is an intervall.*

We now precise the existence's result in the subcritical case :

Theorem 3 *Suppose that* $N \geq 1$*,* Ω^+ *and* $\Omega^- \neq \emptyset$*,* $q < 1^\star$*, and* $\int_\Omega f|\phi|^q < 0$ *for all first eigenfunction* ϕ*. Define*

$$m_q(\lambda) = \inf_{u \in W_0^{1,1}(\Omega),\ \int_\Omega f|u|^q = -1} \{J_\lambda(u)\} = \inf_{u \in BV(\Omega),\ \int_\Omega f|u|^q = -1} \{J_\lambda(u) + \int_{\partial\Omega} |u|\}$$

and

$$p_q(\lambda) = \inf_{u \in W_0^{1,1}(\Omega),\ \int_\Omega f|u|^q = 1} \{J_\lambda(u)\} = \inf_{u \in BV(\Omega),\ \int_\Omega f|u|^q = 1} \{J_\lambda(u) + \int_{\partial\Omega} |u|\}$$

Then, for λ *sufficiently close to* λ_1*,* $\lambda > \lambda_1$*,* $m_q(\lambda)$ *and* $p_q(\lambda)$ *possess each at least one solution, which is, up to a multiplicative constant, a solution to the p.d.e.* eq_λ*. For* $\lambda = \lambda_1$*,* $p_q(\lambda_1)$ *provides, up to a multiplicative constant, a nontrivial solution to* eq_{λ_1}*.*

Remark 4 *We shall denote as $J_{\lambda,r}$ the relaxed form of J_λ, $J_{\lambda,r}(u) = J_\lambda(u) + \int_{\partial\Omega} |u|$ for $u \in BV(\Omega)$. (See[34], [35] for a more general definition of the relaxed formulation).*

Theorem 4 *Suppose that $N \geq 2$, that Ω^+ and $\Omega^- \neq \emptyset$, and that $\int_\Omega f\phi^{1^\star} < 0$ for all first eigenfunction ϕ. Suppose also that $f \in \mathcal{C}(\overline{\Omega})$. Define*

$$m(\lambda) = \inf_{u \in W_0^{1,1}(\Omega),\ \int_\Omega f|u|^{1^\star} = -1} \{J_\lambda(u)\} = \inf_{u \in BV(\Omega),\ \int_\Omega f|u|^{1^\star} = -1} \{J_\lambda(u) + \int_{\partial\Omega} |u|\}$$

and

$$p(\lambda) = \inf_{u \in W_0^{1,1}(\Omega),\ \int_\Omega f|u|^{1^\star} = 1} \{J_\lambda(u)\} = \inf_{u \in BV(\Omega),\ \int_\Omega f|u|^{1^\star} = 1} \{J_\lambda(u) + \int_{\partial\Omega} |u|\}$$

Then, for $\lambda > \lambda_1$, and for λ sufficiently close to λ_1, $m(\lambda)$ possesses at least one nonnegative solution which is, up to a multiplicative constant, a solution for eq_λ.
Suppose that

$$p(\lambda_1) K(N,1) (\sup_\Omega f)^{\frac{1}{1^\star}} < 1,$$

then, for λ sufficiently close to λ_1, $\lambda \geq \lambda_1$, $p(\lambda)$ possesses a solution which is also, up to a multiplicative constant, a (nontrivial) nonnegative solution to the p.d.e. eq_λ.

The plan of this paper is as follows :

In section 2 we give the results about almost weakly **1** harmonic functions which we shall need here. In the third section we present some existence's result concerning the first eigenvalue and corresponding eigenfunctions, we give some properties of the solutions when they exist, and we exhibit explicit particular solutions in the one dimensional case. The fourth section is devoted to the necessary part, as stated in Theorem 1, and to the proof of Theorem 2.

Section 5 and Section 6 are respectively devoted to the proofs of Theorems 3 and 4.

We end this section by recalling a density result which is classical in the theory of BV functions, a result that we shall frequently use :

Proposition 2 *Suppose that Ω is some bounded domain in $\mathbb{R}^N$ whose boundary is piecewise $\mathcal{C}^1$ and that $u \in BV(\Omega)$. Then, there exists a sequence $u_n \in W_0^{1,1}(\Omega) \cap \mathcal{C}^\infty(\Omega)$, such that*

$$\int_\Omega |u_n - u|^k \to 0$$

for all $k \leq 1^\star$, and

$$\int_\Omega |\nabla u_n| \to \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|.$$

2 Functions weakly almost 1 harmonic

In this section we give some of the properties of almost 1-harmonic functions which will be useful to obtain existence and non existence's results herein.

The properties which follow are all consequences of a technical approximation result which permits in some sense to work with BV functions which are almost 1-harmonic as if they belong to the space $W^{1,1}$.

Proposition 3 *Suppose that Ω is a bounded open set in $\mathbb{R}^N$, which is piecewise $\mathcal{C}^1$ and that $u \in BV(\Omega)$. Suppose that u is almost* **1***-harmonic. Then, there exists a sequence (u_ϵ) in $W_0^{1,1+\epsilon}(\Omega)$, such that $u_\epsilon \rightharpoonup u$ in $BV(\Omega)$ tightly, more precisely*

$$\int_\Omega |u_\epsilon - u|^q \to 0,$$

for all $q < 1^\star$ and

$$\int_\Omega |\nabla u_\epsilon|^{1+\epsilon} \to \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|.$$

Moreover if σ is given by proposition 1 , $\sigma_\epsilon \rightharpoonup \sigma$ in $L^q(\Omega)$ for all $q < \infty$ and $\mathrm{div}(\sigma_\epsilon) - \mathrm{div}\sigma$ tends to zero in $L^N(\Omega)$.

This result permits to prove the results which follow.

Some of the proofs are detailed in [15]. For others, and for convenience of the reader, we have given an outline of their proof in the appendix. It is the case for the L^∞ bound type result (Proposition6) and for the Picone's type inequality (Proposition7).

Proposition 4 *Suppose that u is weakly almost 1-harmonic, then u^+ and u^- are also weakly almost 1-harmonic, and*

$$\sigma(u).\nabla u^+ = |\nabla u^+|$$

$$\sigma(u).\nabla(-u^-) = |\nabla u^-|$$

on $\Omega \cup \partial\Omega$.

Proposition 5 *(Weak Comparison Principle). Suppose that u^1 and u^2 are in $BV(\Omega)$ and almost 1 harmonic in Ω. Suppose also that*

$$-\mathrm{div}(\sigma(u^1)) = f^1 > f^2 = -\mathrm{div}(\sigma(u^2))$$

(in the sense that $f_1(x) > f_2(x)$ for almost every $x \in \Omega$. Then

$$u^1 \geq u^2$$

in Ω.

Proposition 6 *Suppose that $u \in BV(\Omega) \cap L^\infty(\Omega)$ is weakly almost 1-harmonic. Then for all $k \in \mathbb{N}$, $|u|^{k-1}u$ is also almost 1-harmonic and*

$$\sigma(u).\nabla(|u|^{k-1}u) = |\nabla(|u|^{k-1}u)|$$

Proposition 7 *Suppose that u is almost weakly 1 harmonic in Ω, then $u \in L^t(\Omega)$ for all $t < \infty$. If moreover $\mathrm{div}(\sigma(u)) \in L^q(\Omega)$ for some $q > N$, then $u \in L^\infty(\Omega)$.*

We end this enumeration by some technical result which will be a key ingredient for the necessary part in theorem 1

Proposition 8 *Suppose that u and ϕ are nonnegative in $BV(\Omega) \cap L^\infty(\Omega)$, and are almost weakly 1 harmonic in Ω. Then for all $\epsilon > 0$, and for all $k \geq 1$, $q \geq 1$,*

$$(\sigma(u) - \sigma(\phi)).\nabla\left(\frac{\phi^k}{(u+\epsilon)^{q-1}}\right) \leq 0$$

3 On the first eigenvalue for the 1-Laplacian

3.1 The case where $N \geq 2$

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Let us define the first eigenvalue of "minus the 1-Laplacian " as follows

$$\lambda_1 = \inf_{u \in W_0^{1,1}(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u|\} \tag{7}$$

The existence of a minimizer for λ_1 as well as the existence of a p.d.e. satisfied by this minimizer is given in the following

Theorem 5 *One has*

$$\lambda_1 = \inf_{u \in W_0^{1,1}(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u|\} = \inf_{u \in BV(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\} \tag{8}$$

and the infimum on the right hand side above is achieved. Moreover among the minimizers, one of them is nonnegative and satisfies the p.d.e. : $\exists \sigma(\phi) \in L^\infty(\Omega, \mathbb{R}^N)$ *such that*

$$\begin{cases} -\mathrm{div}\sigma(\phi) + (-\lambda_1) = 0 \\ \sigma(\phi).\nabla\phi = |\nabla\phi| \text{ in } \Omega, \quad |\sigma(\phi)|_\infty \leq 1 \\ -\sigma(\phi).\vec{n}\phi = \phi \quad \text{on } \partial\Omega \end{cases} \tag{9}$$

Proof of theorem 5
We prove first that

$$\inf_{u \in W_0^{1,1}(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u|\} = \inf_{u \in BV(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\}$$

One obviously has

$$\inf_{u \in W_0^{1,1}(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u|\} \geq \inf_{u \in BV(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\}$$

For the reverse inequality, let $\delta > 0$ be given, and u_0 be in $BV(\Omega)$, such that $\int_\Omega |u_0| = 1$ and

$$\int_\Omega |\nabla u_0| + \int_{\partial\Omega} |u_0| \leq \inf_{u \in BV(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\} + \delta$$

Using the density result as stated in Proposition 2, there exists a sequence $u_n \in W_0^{1,1}(\Omega)$, which is such that $\int_\Omega |u_n - u_0| \to 0$, and $\int_\Omega |\nabla u_n| \to \int_\Omega |\nabla u_0| + \int_{\partial\Omega} |u_0|$. Taking $v_n = \frac{u_n}{\int_\Omega |u_n|}$, one has a sequence v_n which converges towards u in the sense that $\int_\Omega |\nabla v_n| \to \int_\Omega |\nabla u_0| + \int_{\partial\Omega} |u_0|$.

This implies that

$$\inf_{u\in W_0^{1,1}(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u|\} = \inf_{u\in BV(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\}.$$

We now prove that there exists a minimizer for

$$\inf_{u\in BV(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\}.$$

Let $(u_n)_n$ be a minimizing sequence for this problem. Then, the extension of u_n by zero outside of $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. Hence, one can extract from it a subsequence, still denoted u_n, such that $u_n \rightharpoonup u$ in $BV(\mathbb{R}^N)$ weakly. Obviously $u = 0$ ouside of $\overline{\Omega}$. Using lower semicontinuity, one has

$$\int_{\mathbb{R}^N} |\nabla u| \leq \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n| = \lambda_1$$

By the compactness of the Sobolev embedding from $BV(\Omega)$ into $L^1(\Omega)$, one has $|u|_1 = 1$. Since $u = 0$ outside of $\overline{\Omega}$, one has $\nabla u = (-u)\vec{n}\delta_{\partial\Omega}$ on $\partial\Omega$, and then $\int_{\mathbb{R}^N} |\nabla u| = \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|$ and u is a solution for (8).

In order to prove that u satisfies (9), let us consider the variational formulation

$$\lambda^\epsilon = \inf_{u\in W_0^{1,1+\epsilon}(\Omega), \int_\Omega |u|=1} \{\int_\Omega |\nabla u|^{1+\epsilon}\}.$$

Since $|\nabla|u|| = |\nabla u|$, and $\int_\Omega |u| \geq \int_\Omega u$, one has

$$\lambda^\epsilon = \inf_{u\in W_0^{1,1+\epsilon}(\Omega), \int_\Omega u=1} \{\int_\Omega |\nabla u|^{1+\epsilon}\}.$$

It is clear that $\overline{\lim}_{\epsilon\to 0}\lambda^\epsilon \leq \lambda_1$. On the other hand, the problem defining λ^ϵ possesses a solution u_ϵ which is nonnegative and satisfies the p.d.e.

$$-\mathrm{div}(\sigma_\epsilon(u_\epsilon)) - \lambda^\epsilon = 0$$

where

$$\sigma_\epsilon = |\nabla u_\epsilon|^{\epsilon-1}\nabla u_\epsilon.$$

Since $\int_{\mathbb{R}^N} |\nabla u_\epsilon|^{1+\epsilon} = \lambda^\epsilon$ and $\int_\Omega |u_\epsilon| = 1$, the sequence (u_ϵ) is bounded in $BV(\mathbb{R}^N)$, and by extracting from it a subsequence,- still denoted (u_ϵ)- one gets the existence of $u \in BV(\mathbb{R}^N)$ which is zero outside of $\overline{\Omega}$, and satisfies, by lower semicontinuity

$$\lambda_1 \leq \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| \leq \underline{\lim}_{\epsilon\to 0} \int_\Omega |\nabla u_\epsilon|^{1+\epsilon} = \lim_{\epsilon\to 0}\lambda_\epsilon \leq \lambda_1.$$

This implies that $\int_{\mathbb{R}^N} |\nabla u| = \lim \int_{\mathbb{R}^N} |\nabla u_\epsilon|$. On another hand, one has that σ_ϵ is bounded in every L^k for $k < \infty$, then, it converges for a subsequence, in every $L^k(\Omega)$ weakly, to some $\sigma \in \cap_k L^k$. In fact, $\sigma \in L^\infty$, since $|\sigma|_\infty \leq \limsup |\sigma_\epsilon|_{\frac{1+\epsilon}{\epsilon}} \leq 1$. This implies that $\sigma \in L^\infty(\Omega)$ with a norm less than 1. Passing to the limit in the equation satisfied by u_ϵ one obtains that

$$-\mathrm{div}\sigma - \lambda_1 = 0 \tag{10}$$

There remains to prove that $\sigma.\nabla u = |\nabla u|$ in Ω and $\sigma.\vec{n}(u) = -u$ on $\partial\Omega$. First using Proposition 1, one has $|\sigma.\nabla u| \leq |\nabla u|$. Secondly, multiplying the equation (10) by u and integrating over Ω one obtains

$$\int_\Omega \sigma.\nabla u + \int_{\partial\Omega} \sigma.\vec{n}(-u) - \lambda_1 \int_\Omega u = 0$$

This implies that

$$\int_\Omega \sigma.\nabla u + \int_{\partial\Omega} \sigma.\vec{n}(-u) = \int_{\mathbb{R}^N} |\nabla u| = \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|$$

and then $\sigma.\nabla u = |\nabla u|$ in Ω and $\sigma.\vec{n}(u) = -u$ on $\partial\Omega$. Finally u is a solution for (2) and we have obtained in the same time that $\lambda^\epsilon \to \lambda_1$.

Remark 5 *One could have used duality and convex analysis, as developped by Ekeland and Temam in [20], to find the partial differential equation satisfied by u.*

Remark 6 *One proves in [14] that among the eigenfunctions there exists the characteristic function of some set E. This will be useful when we shall consider particular sets Ω (see the examples proposed below).*

Proposition 9 *Suppose that $\lambda \geq \lambda_1$ and that u is some nonnegative eigenfunction associated to λ, say*

$$\begin{cases} -\mathrm{div}\sigma(u) - \lambda = 0 \\ \sigma(u).\nabla u = |\nabla u| \text{ in } \Omega \cup \partial\Omega \end{cases}$$

Then, $\lambda = \lambda_1$.

Proof

Suppose that $\phi \geq 0$ is some eigenfunction for the first eigenvalue λ_1. Suppose that there exists some $u \geq 0$ which satisfies for $\lambda \geq \lambda_1$

$$\begin{cases} -\mathrm{div}(\sigma(u)) - \lambda = 0 & \text{in } \Omega \\ \sigma.\nabla u = |\nabla u| \text{ in } \Omega, \ \sigma.\vec{n}u = -u & \text{on } \partial\Omega \end{cases}$$

Substracting the equation satisfied by u to the one satisfied by ϕ, multiplying by ϕ and integrating over Ω, one gets

$$\int_\Omega (\sigma(u) - \sigma(\phi)).\nabla\phi + \int_{\partial\Omega} (\sigma(\phi) - \sigma(u)).\vec{n}\phi = (\lambda - \lambda_1) \int_\Omega \phi$$

The two integrals on the left hand side are negative, hence one gets that $(\lambda - \lambda_1) \int_\Omega \phi \leq 0$, which implies that $\lambda = \lambda_1$.

Proposition 10 *Suppose that ϕ is an eigenfunction for the first eigenvalue λ_1, $\phi \geq 0$. Then, $\phi \in L^\infty(\Omega)$ and for all f strictly increasing and $\mathcal{C}^1$, such that $f(0) = 0$, $f(\phi)$ is also an eigenfunction for λ_1.*

Proof : One uses Propositions 6 and 7.

Example

Suppose that $B(0, R)$ is a bounded open ball in $\mathbb{R}^N$. Then, the only eigenfunctions are the constants and $\lambda_1 = \frac{N}{R}$.

Indeed, let $\sigma(x) = -\frac{\vec{x}}{R}$. σ satifies $|\sigma| \leq 1$ in $B(0, R)$, and

$$\sigma.\nabla(cte) = 0$$

inside Ω, and

$$\sigma.\vec{n}(-cte) = |cte|$$

for $|x| = R$. Since $-\mathrm{div}\sigma = \frac{N}{R}$, this proves that the constant are eigenfunctions for the eigenvalue $\frac{N}{R}$. Using Proposition 9, necessarily, $\frac{N}{R}$ is the first eigenvalue. To see that the only eigenfunctions are the constant functions, suppose that ψ is another eigenfunction, and multiply the equation $-\mathrm{div}(\sigma) = \frac{N}{R}$ by ψ, with σ defined above. Then one obtains that

$$\int_B \sigma.\nabla\psi + \int_{\partial B} \psi = \frac{N}{R} \int_B \psi$$

Since ψ is a solution,

$$\frac{N}{R} \int_B \psi = \int_B |\nabla\psi| + \int_{\partial B} \psi$$

and then

$$\sigma.\nabla\psi = |\nabla\psi|$$

in Ω. This implies since $|\sigma| < 1$ inside Ω, that $\nabla\psi = 0$ in Ω and then $\psi = cte$. Let us observe that the same result holds with a crown $B(0, R_2) - B(0, R_1)$ in place of a ball. The fact that constant functions are eigenfunctions is linked to of the euclidian structure of the ball and some of related sets : For example, the square $C =]-1, 1[^2$ does not admit the constant functions as eigenfunctions for the first eigenvalue. One can prove that $\lambda \leq \frac{7+\frac{\pi}{4}}{3+\frac{3}{4}+\frac{\pi}{16}} < 2 = \frac{|\partial C|}{|C|}$. Indeed, let us consider the set $E = C - C_1$ where C_1 is

$$C_1 = \{-1 \leq x \leq \frac{-1}{2},\ y \geq \frac{1}{2},\ (x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 \geq \frac{1}{4}\}$$

One can compute

$$|\partial E| = 7 + \frac{\pi}{4}$$

$$|E| = 3 + \frac{3}{4} + \frac{\pi}{16}$$

and if one takes u to be the characteristic function of E, $u \in BV(C)$ with

$$\frac{\int_C |\nabla u|}{\int_C |u|} = \frac{|\partial E|}{|E|} = \frac{7+\frac{\pi}{4}}{3+\frac{3}{4}+\frac{\pi}{16}}$$

One can prove with similar arguments that for all rectangles eigenfunctions cannot be constant.
One can also see the more complete results in [3] and [4].

We end this section by establishing a condition which permits to ensure that $\lambda_1 = \frac{|\partial\Omega|}{|\Omega|}$.

Proposition 11 *Suppose that there exists some eigenfunction ϕ for the first eigenvalue λ_1 which satisfies $\phi > 0$ almost everywhere on $\partial\Omega$. Then $\lambda_1 = \frac{|\partial\Omega|}{|\Omega|}$ and the constants are eigenfunctions for the first eigenvalue.*

Proof : Since $\phi > 0$ on $\partial\Omega$, the equation $\sigma.\vec{n}(-\phi) = \phi$ almost everywhere on $\partial\Omega$ implies that $\sigma.\vec{n} = -1$ almost everywhere on $\partial\Omega$. Let us observe then that $\phi + K$ is also an eigenfunction for every constant $K > 0$. Indeed $\sigma(\phi + K) = \sigma(\phi)$ satisfies

$$\sigma(\phi + K).\nabla(\phi) = |\nabla\phi| \text{ in } \Omega$$

and

$$\sigma(\phi).\vec{n}(-(\phi + K)) = \phi + K$$

on the boundary. Then, one has

$$\begin{aligned}
\lambda_1 \int_\Omega (\phi + K) &= \int_\Omega |\nabla(\phi + K)| + \int_{\partial\Omega} |\phi + K| \\
&= \int_\Omega |\nabla\phi| + \int_{\partial\Omega} |\phi| + K|\partial\Omega|
\end{aligned}$$

This implies, using the fact that ϕ is an eigenfunction, that

$$\lambda_1 K|\Omega| = K|\partial\Omega|$$

Once one knows that $\lambda_1 = \frac{|\partial\Omega|}{|\Omega|}$, it is obvious that the constant are eigenfunctions.

3.2 The one dimensional case

Proposition 12 *Suppose that $\Omega =]0,1[$. Then, the first eigenvalue equals 2. The only eigenfunctions for the first eigenvalue are the constants.*

Proof of Proposition 12

We begin to prove that $\lambda_1 = 2$. For that aim, suppose that $u \in W_0^{1,1}(]0,1[)$, then for every $x \in]0,1[$

$$u(x) = \int_0^x u'(t)dt$$

$$u(x) = -\int_x^1 u'(t)dt$$

From this one gets that for every $x \in]0,1[$

$$2|u(x)| \leq \int_0^1 |u'(t)|dt$$

and integrating over $]0,1[$

$$2\int_0^1 |u| \leq \int_0^1 |u'|$$

This implies that $\lambda_1 \geq 2$. For the reverse inequality take the sequence

$$u_n = nx\chi_{[0,\frac{1}{n}]} + \chi_{[\frac{1}{n},1-\frac{1}{n}]} + n(1-x)\chi_{[1-\frac{1}{n},1]}$$

One has

$$u'_n = n\chi_{[0,\frac{1}{n}]} - n\chi_{[1-\frac{1}{n},1]},$$

hence

$$\int_0^1 |u'_n| = 2$$

Since $\int_0^1 |u_n| = 1 - \frac{1}{n} \to 1$, one gets the result.

We now prove that the only eigenfunctions for $\lambda_1 (= 2)$ are constant functions.

An eigenfunction ϕ must satisfy

$$-\sigma' = 2$$

$$\phi'\sigma = |\phi'| \text{ in }]0,1[$$

$$-\sigma(1)\phi(1) = |\phi(1)|$$

$$\sigma(0)\phi(0) = |\phi(0)|$$

The equation satisfied by σ implies that there exists $\alpha \in [-1,1]$, such that

$$\sigma(x) = \alpha - 2x$$

Since σ has values in $[-1,1]$, necessarily $\alpha = 1$. As a consequence $\sigma(x) \in]-1,1[$ for $x \in]0,1[$, hence $\phi' = 0$ in $]0,1[$. Finally $\phi = cte$ and it is easy to verify that every constant is a solution of the relaxed problem

$$\inf_{\phi \in BV(]0,1[),\ \int_\Omega |\phi| = 1} \left\{ \int_0^1 |\phi'| + |\phi(0)| + |\phi(1)| \right\}$$

4 Properties of the solutions. Necessary conditions on f and the first eigenfunctions

Let f be some continuous function on $\overline{\Omega}$.

Proposition 13 *Suppose that u is a solution for eq_λ for $\lambda \geq \lambda_1$, and that $1 < q \leq N+1$. Then $u \in L^\infty(\Omega)$.*

Using first part in Proposition 7, one gets that $u \in L^k$ for all $k < \infty$. Then $-\mathrm{div}\sigma(u) = \lambda + fu^{q-1} \in L^s$ for all s, hence using once more Proposition 7 in its second part, one gets $u \in L^\infty$.

Theorem 6 *Suppose that for $\lambda > \lambda_1$ there exists a nonnegative solution to the p.d.e. eq_λ. Then, if ϕ is some nonnegative eigenfunction for the eigenvalue λ_1, $\int_\Omega f\phi^q < 0$.*

Suppose that $\lambda = \lambda_1$, and that there exists a nontrivial solution to eq_{λ_1}. then $\Omega^+ \neq \emptyset$.

Proof

Let $\epsilon > 0$ be given. Substract the equation satisfied by ϕ from the one satisfied by u, and multiply by $\frac{\phi^q}{(u+\epsilon)^{q-1}}$ which belongs to BV (since $\phi \in L^\infty$). Using Proposition 8 one has

$$(\sigma(u) - \sigma(\phi)).\nabla\left(\frac{\phi^q}{(u+\epsilon)^{q-1}}\right) \leq 0$$

and since

$$\frac{\phi^q}{(u+\epsilon)^{q-1}} + (\sigma(u).\vec{n})\frac{\phi^q}{(u+\epsilon)^{q-1}} \geq 0 \text{ on } \partial\Omega$$

one obtains after integrating over Ω and using generalized Green's formula in 1 :

$$(\lambda - \lambda_1)\int_\Omega \frac{\phi^q}{(u+\epsilon)^{q-1}} \leq -\int_\Omega f\frac{u^{q-1}}{(u+\epsilon)^{q-1}}\phi^q \tag{11}$$

Let us observe that the right hand side of (11) converges when ϵ goes to zero, (using the dominated convergence theorem) to the limit $-\int_{x,u(x)>0} f\phi^q$. On the other hand, the left hand side is increasing and nonnegative, hence it is convergent. This implies that $\dfrac{\phi^q}{u^{q-1}}$ is integrable. Let us denote by α the L^1 function such that $\phi^q = \alpha u^{q-1}$. Then

$$\int_{x,u(x)>0} f\phi^q = \int_{x,u(x)>0} f\alpha u^{q-1} = \int_\Omega f\alpha u^{q-1} = \int_\Omega f\phi^q$$

(In the previous inequalities, one has used $q > 1$). One has finally obtained that any nonnegative eigenfunction ϕ for the first eigenvalue λ_1 satisfies $\int_\Omega f\phi^q < 0$.

Suppose now that $\lambda = \lambda_1$, let u be a nonnegative, nontrivial solution to eq_{λ_1}. Multiplying the equation by u, one gets $\int_\Omega f u^q \geq 0$ and since it cannot be zero because u is not an eigenfunction, one has $\int_\Omega f u^q > 0$. In particular $\Omega^+ \neq \emptyset$.

Let us note that one can have solutions for $\lambda = \lambda_1$ and $\int_\Omega f\phi^q = 0$:

Suppose that $k > 1$, that $B(0,k)$ is a bounded open ball in $\mathbb{R}^N$. We have already seen that the eigenfunctions are the constants and $\lambda_1 = \frac{N}{k}$. Let us define σ as

$$\sigma(x) = \begin{cases} -\vec{x} \text{ if } |\vec{x}| \leq 1 \\ -\frac{\vec{x}}{|x|} \text{ if } |x| \in [1,k] \end{cases}$$

σ satifies $|\sigma| \leq 1$, and

$$\sigma.\nabla(cte) = 0$$

inside Ω, and

$$\sigma\vec{n}(-cte) = |cte|$$

for $|x| = k$. One has

$$-\mathrm{div}(\sigma) = \frac{N}{k} + f$$

where

$$f = \begin{cases} N\left(1 - \frac{1}{k}\right) \text{ if } |x| \leq 1 \\ \frac{N-1}{|x|} - \frac{N}{k} \; 1 \leq |x| \leq k \end{cases}$$

One can check that $\int_{B(0,k)} f = 0$, and the previous equations prove that $u = cte$ is solution.

Proposition 14 *Let N be an integer, $N \geq 2$. Suppose that ϕ is some nonnegative eigenfunction for λ_1 and that u is some nonnegative solution for eq_λ, $\lambda > \lambda_1$. Then, there exists $\alpha \in L^\infty(\Omega)$ such that $\phi = \alpha u^{q-1}$.*

Proof of Proposition 14.

We begin as in the proof of Theorem 6 : Let us multiply the equation by $\dfrac{\phi}{(u+\epsilon)^{q-1}}$ and integrate over Ω. One obtains

$$\begin{aligned} \int_\Omega (\sigma(u) - \sigma(\phi)).\nabla\left(\frac{\phi}{(u+\epsilon)^{q-1}}\right) \quad &+ \quad \int_{\partial\Omega} (-\sigma(u) + \sigma(\phi)).\vec{n}\left(\frac{\phi}{(u+\epsilon)^{q-1}}\right) + \\ &+ \quad (\lambda_1 - \lambda)\int_\Omega \frac{\phi}{(u+\epsilon)^{q-1}} \\ &= \quad \int_\Omega f\phi\frac{u^{q-1}}{(u+\epsilon)^{q-1}} \end{aligned}$$

the right hand side tends to $\int_{x,u(x)>0} f\phi$, and using the negativity of $\int_\Omega (\sigma(u)-\sigma(\phi)).\nabla\left(\frac{\phi}{(u+\epsilon)^{q-1}}\right)$ of $\int_{\partial\Omega}(-\sigma(u)+\sigma(\phi)).\vec{n}\left(\frac{\phi}{(u+\epsilon)^{q-1}}\right)$ and the negativity of $(\lambda_1-\lambda)\int_\Omega \frac{\phi}{(u+\epsilon)^{q-1}}$, one gets that $\lim_{\epsilon\to 0}\int_\Omega \frac{\phi}{(u+\epsilon)^{q-1}}$ exists. This implies that according to the monotone convergence theorem of Lebesgue, the function $\frac{\phi}{(u)^{q-1}}$ is in $L^1(\Omega)$.

Suppose that we have proved that for $j\leq k$, $\frac{\phi^j}{u^{(q-1)j}} = \alpha_j \in L^1(\Omega)$, and let us multiply the equation in u by $\frac{\phi^{k+1}}{(u+\epsilon)^{(q-1)(k+1)}}$. One obtains

$$(\lambda-\lambda_1)\int_\Omega \frac{\phi^{k+1}}{(u+\epsilon)^{(q-1)(k+1)}} \leq -\int_\Omega f u^{q-1}\frac{\phi^{k+1}}{(u+\epsilon)^{(q-1)(k+1)}}$$

The right hand side tends to the finite limit $\int_\Omega f\phi\alpha_k$ when $\epsilon\to 0$. This implies that $\frac{\phi^{k+1}}{u^{(q-1)(k+1)}} = \alpha_{k+1}\in L^1(\Omega)$ and

$$\int_\Omega \alpha_{k+1} \leq \frac{|f|_\infty|\phi|_\infty}{\lambda-\lambda_1}\int_\Omega \alpha_k \leq C\left(\int_\Omega \alpha_{k+1}\right)^{\frac{k}{k+1}}|\Omega|^{\frac{1}{k}}$$

From this one obtains that $(\int_\Omega \alpha_{k+1})^{\frac{1}{k+1}} \leq C$ and then $\frac{\phi}{u^{q-1}}\in L^\infty(\Omega)$. More precisely

$$\left|\frac{\phi}{u^{q-1}}\right|_\infty \leq \frac{|f|_\infty|\phi|_\infty}{(\lambda-\lambda_1)}.$$

4.1 Properties of the solutions in the one dimensional case. Some explicit examples

Proposition 15 *Suppose that $\Omega =]0,1[$, and that u is some nonnegative solution for eq_λ, $\lambda>\lambda_1$. Then, u is bounded from below by a positive constant.*

Proof of Proposition 15

Let us substract the equation satisfied by ϕ to the equation satisfied by u, multiply by $\frac{1}{(u+\epsilon)^{q-1}}$ and integrate over $]0,1[$ One obtains

$$\begin{aligned}\int_0^1 (\sigma(u)-\sigma(\phi))\left(\frac{1}{(u+\epsilon)^{q-1}}\right)' &+ \frac{-\sigma(u)(1)-1}{(u(1)+\epsilon)^{q-1}} + \frac{\sigma(u)(0)-1}{(u(0)+\epsilon)^{q-1}} + \\ &+ (\lambda_1-\lambda)\int_0^1 \frac{1}{(u+\epsilon)^{q-1}} \\ &= \int_0^1 f\frac{u^{q-1}}{(u+\epsilon)^{q-1}}.\end{aligned}$$

Using Theorem 6 with $\phi = 1$, one sees that the left hand side is the sum of three negative quantities, the right hand side tends to a finite limit $\int_{x,u(x)>0} f$, and then, using the Lebesgue's monotone convergence theorem, one obtains that $\int_0^1 \frac{1}{(u+\epsilon)^{q-1}}$ tends to a finite limit, hence $\frac{1}{u^{q-1}} \in L^1(]0,1[)$.

Suppose that one has proved that for $k \in \mathbb{N}$, $u^{-k(q-1)} \in L^1$. Mutiplying the substracted equation by $\frac{1}{(u+\epsilon)^{k(q-1)}}$, one gets that

$$(\lambda-\lambda_1)\int_0^1 \frac{1}{(u+\epsilon)^{k(q-1)}} \leq -\int_0^1 f\frac{u^{q-1}}{(u+\epsilon)^{k(q-1)}} \leq |f|_\infty |u+\epsilon|^{k-1}_{L^{(k-1)\epsilon}}.$$

From this one gets that $u^{-k(q-1)} \in L^1$, finally

$$|u^{-q+1}|_k \leq \frac{|f|_\infty}{|\lambda-\lambda_1|}.$$

This implies that $\frac{1}{u} \in L^\infty$, finally

$$u \geq \left(\frac{\lambda-\lambda_1}{|f|_\infty}\right)^{\frac{1}{(q-1)}}$$

Examples:

As we shall see in the fifth section, one is able to prove that when q is subcritical and for $\lambda = \lambda_1$, eq_λ possesses at least one solution, and for $\lambda > \lambda_1$ and λ sufficiently close to λ_1, there are at least two solutions . We present here particular solutions in the following case : $\Omega =]0,1[$, $\lambda = \lambda_1 (= 2)$. In a second time, we present an explicit resolution in the case $\lambda > \lambda_1$, for which we exhibit two solutions.

Let $\Omega =]0,1[$, $\lambda = \lambda_1 (= 2)$, $q < \infty$: Assume that α is some real in $]0,1[$, and

$$f = \begin{cases} c_1 & \text{on } [0, \alpha[\\ c_2 & \text{on } [\alpha, 1] \end{cases}$$

where $c_1 > 0$, $c_2 < 0$ and

$$\int_0^1 f = c_1 \alpha + (1-\alpha) c_2 < 0$$

Let us define the function

$$u = \begin{cases} \left(\dfrac{2(1-\alpha)}{\alpha c_1}\right)^{\frac{1}{q-1}} & \text{on} \quad]0, \alpha[\\ \left(\dfrac{-2}{c_2}\right)^{\frac{1}{q-1}} & \text{on} \quad [\alpha, 1[\end{cases}$$

One has

$$u' = (\left(\frac{-2}{c_2}\right)^{\frac{1}{q-1}} - \left(\frac{2(1-\alpha)}{\alpha c_1}\right)^{\frac{1}{q-1}}) \delta_\alpha = (u_2 - u_1)\delta_\alpha$$

where δ_α denotes the Dirac measure on the point α. Then, u is a solution for the equation

$$-\sigma' = 2 + f u^{q-1}$$

$$\sigma u' = |u'| \text{ on }]0,1[$$

Indeed σ defined by

$$\sigma = \begin{cases} 1 - \dfrac{2}{\alpha} x & \text{on } [0, \alpha] \\ -1 & \text{on } [\alpha, 1[\end{cases}$$

satifies $\sigma(\alpha) = -1$, and then,

$$\sigma u' = \sigma(\alpha)(u_2 - u_1)\delta_\alpha = |u_2 - u_1| \delta_\alpha$$

since $u_2 < u_1$. Moreover the boundary conditions are satisfied, since $\sigma(0) = 1$ and $\sigma(1) = -1$ imply

$$u_1 = \sigma(0) u_1$$

and

$$u_2 = \sigma(1)(-u_2)$$

Suppose now that $c_1 < 0$, $c_2 > 0$ and $c_1\alpha + (1-\alpha)c_2 < 0$. A solution is given by

$$u = \begin{cases} \left(\dfrac{-2}{c_1}\right)^{\frac{1}{q-1}} & \text{on} \quad [0, \alpha[\\ \left(\dfrac{2\alpha}{(1-\alpha)c_2}\right)^{\frac{1}{q-1}} & \text{on} \quad [\alpha, 1[\end{cases}$$

Indeed, σ defined by

$$\sigma = \begin{cases} 1 & \text{on } [0, \alpha] \\ -1 + \frac{2}{1-\alpha}(1-x) & \text{on } [\alpha, 1] \end{cases}$$

is a solution.

Remark 7 *As we pointed out in the proof of Theorem 1, the condition $\int_0^1 f < 0$ is not necessary. Suppose that*

$$\int_0^1 f = c_1\alpha + (1-\alpha)c_2 = 0$$

with $c_1 > 0$, $c_2 < 0$. Then, the constant

$$u = \left(\frac{2(1-\alpha)}{\alpha c_1}\right)^{\frac{1}{q-1}}$$

is a solution for $\lambda = \lambda_1$, since σ defined by

$$\sigma = \begin{cases} 1 - \dfrac{2}{\alpha}x & \text{on } [0, \alpha] \\ -1 & \text{on } [\alpha, 1[\end{cases}$$

satisfies $\sigma.u' = 0 = |u'|$ and $\sigma(1) = -1$, and $\sigma(0) = 1$. This proves that the condition $\int_0^1 f \leq 0$ is optimal when $\lambda = \lambda_1$.

We now assume that $\lambda > \lambda_1 = 2$, and exhibit two solutions for λ sufficiently close to λ_1. Suppose that

$$f = \begin{cases} c_1 & \text{on } [0, \alpha[\\ c_2 & \text{on } [\alpha, 1] \end{cases}$$

where c_1 and c_2 satisfy $c_1 > 0$, $c_2 < 0$, and

$$c_1\alpha + (1-\alpha)c_2 < 0$$

Let us define $\epsilon = \lambda - 2$, and assume that ϵ is small enough in order that

$$\epsilon < \frac{2}{\alpha} - 2$$

and

$$\epsilon < \frac{2}{\alpha} \frac{\int_0^1 f}{(c_2 - c_1)}$$

Define

$$u = \left(\frac{-\epsilon}{\int_0^1 f} \right)^{\frac{1}{q-1}}$$

Then we check that u is a solution. Indeed

$$\sigma(x) = \begin{cases} 1 - (2 + \epsilon + c_1 u^{q-1})x & \text{on} \quad]0, \alpha[\\ -1 + (1 - x)(2 + \epsilon + c_2 u^{q-1}) & \text{on} \quad]\alpha, 1[\end{cases}$$

is continuous, has values in $] - 1, 1[$, for $x \in]0, 1[$, hence

$$\sigma u' = |u'|$$

in $]0, 1[$, and $\sigma(1) = -1$, $\sigma(0) = 1$ imply that the boundary conditions are satisfied. Finally by construction,

$$-\sigma' - (2 + \epsilon) = f u^{q-1}$$

We now present a solution such that $\int_0^1 f u^q > 0$. Suppose that $c_1 > 0$, $c_2 < 0$, $c_1 \alpha + (1 - \alpha) c_2 < 0$, and take $\epsilon < \dfrac{2(1-\alpha)}{\alpha}$, $\epsilon < \dfrac{2 \int_0^1 f}{\alpha(c_2 - c_1)}$, then u defined by

$$u = \begin{cases} \left(\dfrac{\frac{2}{\alpha} - 2 - \epsilon}{c_1} \right)^{\frac{1}{q-1}} & \text{on } [0, \alpha[\\ \left(\dfrac{-(2 + \epsilon)}{c_2} \right)^{\frac{1}{q-1}} & \text{on } [\alpha, 1[\end{cases}$$

solves the problem

5 Non existence results

Theorem 7 *Let us suppose that λ is large. Then, there is no solution to eq_λ.*

Proof
Let B be an euclidian ball on which $f > 0$, and let $\mu^\star$ and ψ be solutions of the eigenvalue problem

$$\mu^\star = \inf_{u \in BV(B), \int_B |u|=1} \{ \int_{B \cup \partial B} |\nabla u| \},$$

One knows by the results in section 3 that $\mu^\star = \frac{|\partial B|}{|B|}$ and that the positive constant are minimizers for $\mu^\star$. Suppose that there exists a solution u for eq_λ with $\lambda > \mu^\star$. Then

$$-\mathrm{div}(\sigma(u)) - \lambda \geq 0 \text{ in } B,$$

with

$$\sigma(u).\nabla u = |\nabla u| \text{ on } \overline{\Omega}.$$

Integrating this over B, one obtains

$$-\int_{\partial B} \sigma(u).\vec{n} \geq \lambda |B|,$$

which implies since $|\sigma(u).\vec{n}| \leq 1$ that

$$\lambda \leq \frac{|\partial B|}{|B|},$$

a contradiction.

Remark 8 *This has also the following consequence : If eq_λ possesses a solution then $\lambda \leq \inf_{\{B,\ f>0 \text{ on } B\}} \frac{|\partial B|}{|B|}$.*

We now prove that the set of λ for which eq_λ possesses a solution is an intervall. To prove this kind of result, one usually uses an argument of sub and supersolutions, which is a consequence of the following result : We suppose here that

$$f(x, u) = f(x)u^{q-1} + \lambda,$$

we consider the equation

$$\begin{cases} -\mathrm{div}(\sigma) = f(x, u) \\ \sigma.\nabla u = |\nabla u| \text{ on } \Omega \cup \partial\Omega \end{cases} \tag{12}$$

with $q > 1$, $\lambda \geq \lambda_1$ with λ_1 the first eigenvalue for minus the "1-Laplacian" on Ω.

Theorem 8 *Suppose that there exists $\overline{u}$ and $\underline{u}$ which are respectively super and subsolutions of (12). Suppose in addition that there exists $C > 0$ such that*

$$0 \le \underline{u} \le \overline{u} \le C$$

Then, there exists a solution to (12) which is such that $\underline{u} \le u \le \overline{u}$.

The proof of this result can be found in [14]. It uses essentially perturbed variational problems and comparison of solutions, as it is done in [23].

To apply this theorem we need to observe first that if there exists a solution $u_{\lambda'}$ for $eq_{\lambda'}$, and $\lambda' > \lambda_1$, then, there are subsolutions for eq_λ. Indeed, for $\epsilon > 0$ small, if ϕ is some nonnegative first eigenfunction, $\epsilon\phi$ is a subsolution for eq_λ, since one has $\sigma(\epsilon\phi) = \sigma(\phi)$, and then

$$-\text{div}\sigma(\epsilon\phi) - \lambda = \lambda_1 - \lambda \le f(\epsilon\phi)^{q-1}$$

for ϵ small enough, using the fact that f is bounded.

On the other hand $u_{\lambda'}$ is a supersolution for eq_λ. We prove then that $u_{\lambda'}$ is bounded from below by some subsolution.

Suppose that $N = 1$, then, using Proposition 15, $u_{\lambda'} > \dfrac{\lambda' - \lambda_1}{|f|_\infty}$ and then, for all constant (eigenfunction) ϕ, there exists $\epsilon > 0$ such that $u_{\lambda'} \ge \epsilon\phi$. Suppose that $N \ge 2$, then $\frac{1}{q-1} \ge 1$ and $\phi^{\frac{1}{q-1}}$ is an eigenfunction. Choosing $\epsilon < \frac{1}{|\alpha|_\infty^{\frac{1}{q-1}}}$ where $\phi = \alpha u_{\lambda'}^{q-1}$, and α is given by Proposition (14), one has

$$u_{\lambda'} \ge \epsilon\phi^{\frac{1}{q-1}}.$$

6 Some existence's results in the subcritical case

In all that section we assume that $q < 1^\star = \frac{N}{N-1}$.

Theorem 9 *Suppose that Ω^+ and Ω^- are $\neq \emptyset$ and that every nonnegative eigenfunction ϕ for the first eigenvalue λ_1 satisfies $\int_\Omega f\phi^q < 0$. Define*

$$\lambda_q^\star = \inf_{u \in BV(\Omega),\ |u|_1 = 1,\ \int_\Omega f|u|^q = 0} \{\int_\Omega (|\nabla u| - \lambda_1|u|) + \int_{\partial\Omega} |u|\}$$

and for $\lambda \ge \lambda_1$, the two infima

$$m_q(\lambda) = \inf_{u \in BV(\Omega),\ \int_\Omega f|u|^q = -1} \{\int_\Omega (|\nabla u| - \lambda|u|) + \int_{\partial\Omega} |u|\}$$

and

$$p_q(\lambda) = \inf_{u \in BV(\Omega),\ \int_\Omega f|u|^q = 1} \{\int_\Omega (|\nabla u| - \lambda |u|) + \int_{\partial\Omega} |u|\}.$$

Then $\lambda_q^\star$ *is* > 0 *and achieved, and for* $\lambda \in [\lambda_1, \lambda_1 + \lambda_q^\star]$, $m_q(\lambda) < 0$ *and is achieved,* $m_q(\lambda_1) = 0$, $p_q(\lambda) > 0$ *and is achieved.*

Remark 9 *Of course, using the density result in Proposition (2) the infimum defining* $m_q(\lambda)$, $p_q(\lambda)$ *and* $\lambda_q^\star$ *can be taken on* $W_0^{1,1}(\Omega)$ *in place of* $BV(\Omega)$.

Proof

We begin to prove that $\lambda_q^\star$ is positive and achieved. Suppose that (u_n) is a minimizing sequence defining $\lambda_q^\star$, $u_n \in W_0^{1,1}(\Omega)$. One can assume that $u_n \geq 0$. Then, the extension of u_n by zero outside $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. Extracting from it a subsequence, still denoted u_n for simplicity, one obtains that $u_n \rightharpoonup u$ in $BV(\mathbb{R}^N)$ weakly. u is nonnegative and is zero outside $\overline{\Omega}$. Using lower semicontinuity for the weak topology, one has

$$\int_{\mathbb{R}^N} |\nabla u| = \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| \leq \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n| = \int_\Omega |\nabla u_n|.$$

Since q is subcritical, the embedding of $BV(\Omega)$ into $L^q(\Omega)$ is compact and then $\int_\Omega f u^q = 0$. Suppose that $\lambda_q^\star = 0$, then, one would get u a nonnegative eigenfunction for the eigenvalue λ_1 such that $\int_\Omega f|u|^q = 0$, a contradiction. Finally $\lambda_q^\star > 0$. Suppose now that $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^\star[$. Let us prove that $m_q(\lambda) > -\infty$. If not, one would have a sequence $u_n \geq 0$, $u_n \in W_0^{1,1}(\Omega)$, such that $|u_n|_1 \to \infty$, and $\int_\Omega |\nabla u_n| - \lambda |u_n|_1 \to -\infty$. Let us extend u_n by zero outside $\overline{\Omega}$, and define $w_n = \frac{u_n}{|u_n|_1}$. w_n is a nonnegative bounded sequence in $BV(\mathbb{R}^N)$, which is zero outside of $\overline{\Omega}$. Extracting from it a subsequence, one gets that it converges to some $w \in BV(\mathbb{R}^N)$ weakly. w is zero outside $\overline{\Omega}$, satisfies $|w|_1 = 1$ and since q is subcritical $\int_\Omega f w^q = 0$. In the same time by lower semicontinuity one has

$$\int_\Omega |\nabla w| + \int_{\partial\Omega} |w| - \lambda \int_\Omega |w| = \int_{\mathbb{R}^N} |\nabla w| - \lambda \leq 0,$$

which contradicts the assumption $\lambda \in [\lambda_1, \lambda_1 + \lambda_q^\star[$.

Let us observe that the same arguments prove that every minimizing sequence for $m_q(\lambda)$ is bounded in $BV(\mathbb{R}^N)$. Extracting from it a subsequence, and using lower semicontinuity for the vague topology, one gets by passing to the limit, the existence of a minimizer $u_q(\lambda)$ for $m_q(\lambda)$. Observing that if ϕ is some nonnegative eigenfunction for the first eigenvalue, $\frac{\phi}{(-\int_\Omega f\phi^q)^{\frac{1}{q}}}$ is admissible for $m_q(\lambda)$ one gets that $m_q(\lambda) < 0$ as soon as $\lambda > \lambda_1$. Let us observe that $m_q(\lambda_1) = 0$.

We act for $p_q(\lambda)$ as for $m_q(\lambda)$, more precisely we are able to prove in the same manner that $p_q(\lambda) > -\infty$ and it is achieved.

We want to prove that any nonnegative minimizer for $m_q(\lambda)$ satisfies the p.d.e.

$$-\mathrm{div}(\sigma(u)) - \lambda = -m_q(\lambda) f u^{q-1}, \tag{13}$$

where $|\sigma|_{L^\infty(\Omega)} \leq 1$, $\sigma.\nabla u = |\nabla u|$ in Ω, $\sigma.\vec{n}u = -u$ on $\partial\Omega$. For that aim, let us introduce for q fixed, for $\epsilon > 0$ given and small, the following variational problem

$$m_\epsilon(\lambda) = \inf_{u \in W_0^{1,1+\epsilon}(\Omega), \int_\Omega f|u|^q = -1} \{\int_\Omega |\nabla u|^{1+\epsilon} - \lambda \int_\Omega |u|\}.$$

This problem possesses a minimizer, since the functional involved is coercive on the space $W_0^{1,1+\epsilon}(\Omega)$. Let us observe that $\lim_{\epsilon\to 0} m_\epsilon(\lambda) = m_q(\lambda)$. Indeed, let $\delta > 0$ and $\epsilon_0 > 0$ be given and u be in $W_0^{1,1+\epsilon_0}(\Omega)$, , such that $\int_\Omega f|u|^q = -1$ and

$$\int_\Omega |\nabla u| - \lambda \int_\Omega |u| \leq m_q(\lambda) + \delta.$$

Then $\int_\Omega |\nabla u|^{1+\epsilon} \to \int_\Omega |\nabla u|$ when $\epsilon \to 0$, hence $\overline{\lim}_{\epsilon\to 0} m_\epsilon(\lambda) \leq m_q(\lambda) + \delta$. For the reverse inequality, let u_ϵ be a nonnegative minimizer for $m_\epsilon(\lambda)$. Suppose for a while that one has proved that u_ϵ is bounded in $W^{1,1+\epsilon}(\Omega)$. Then, denoting still by u_ϵ the extension of u_ϵ by zero outside of $\overline{\Omega}$, and extracting from it a subsequence, one has after passing to the limit the existence of a nonnegative $u \in BV(\mathbb{R}^N)$, which is zero outside of $\overline{\Omega}$, and satisfies $\int_\Omega f|u|^q = -1$, (since q is subcritical and using the compactness of the Sobolev embedding from $BV(\Omega)$ into $L^q(\Omega)$ for Ω bounded). By lower semicontinuity for the weak topology of the BV norm in $\mathbb{R}^N$, one has

$$\int_\Omega |\nabla u| + \int_{\partial\Omega} |u| = \int_{\mathbb{R}^N} |\nabla u| \leq \underline{\lim} \int_{\mathbb{R}^N} |\nabla u_\epsilon|^{1+\epsilon} = \underline{\lim}_{\epsilon\to 0} \int_\Omega |\nabla u_\epsilon|^{1+\epsilon}.$$

This implies that

$$m_q(\lambda) \leq \underline{\lim}_{\epsilon\to 0} m_\epsilon(\lambda).$$

Let u_ϵ be a nonnegative minimizer for the problem defining $m_\epsilon(\lambda)$. It satisfies the following p.d.e.

$$-\text{div}(\sigma_\epsilon(u_\epsilon)) - \frac{\lambda}{1+\epsilon} = -m'_\epsilon(\lambda) f u_\epsilon^{q-1}, \tag{14}$$

with $m'_\epsilon(\lambda) = m_\epsilon(\lambda) + \frac{\epsilon\lambda}{1+\epsilon}\int_\Omega u_\epsilon^{1+\epsilon}$. Let us note that $\sigma_\epsilon = |\nabla u_\epsilon|^{\epsilon-1}\nabla u_\epsilon$ is bounded in $L^{\frac{1+\epsilon}{\epsilon}}(\Omega)$. Then, it converges, up to a subsequence, in every $L^k(\Omega)$ weakly for all $k < \infty$, towards some $\sigma \in \cap_k L^k(\Omega)$. Using $\limsup |\sigma_\epsilon|_{\frac{1+\epsilon}{\epsilon}} \leq 1$, one gets that $\sigma \in L^\infty$ with some L^∞ norm less than 1. On the other hand, passing to the limit in (14) one has

$$-\text{div}\sigma - \lambda = -m_q(\lambda) f u^{q-1}.$$

Multiplying this last equation by u and integrating one obtains

$$\begin{aligned} \int_\Omega (\sigma\cdot\nabla u - \lambda u) + \int_{\partial\Omega}(-\sigma\cdot\vec{n}u) &= -m_q(\lambda)\int_\Omega f u^q \\ &= m_q(\lambda) = \int_\Omega (|\nabla u| - \lambda u) + \int_{\partial\Omega}|u|. \end{aligned}$$

From this one gets that $\sigma\cdot\nabla u = |\nabla u|$ and $\sigma\cdot\vec{n}u = -u$ on $\partial\Omega$. Finally u satisfies (13).

There remains to prove that the sequence u_ϵ is bounded in $BV(\mathbb{R}^N)$. It suffices to establish that is is bounded in $L^1(\Omega)$. Suppose by contradiction that $\int_\Omega u_\epsilon \to \infty$. Then, for ϵ small enough $\int_\Omega u_\epsilon > 1$. and then $(\int_\Omega u_\epsilon)^{1+\epsilon} \geq \int_\Omega u_\epsilon$. By defining $w_\epsilon = \frac{u_\epsilon}{\int_\Omega u_\epsilon}$, one has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_\epsilon|^{1+\epsilon} - \lambda &= \int_{\mathbb{R}^N} \frac{|\nabla u_\epsilon|^{1+\epsilon}}{(\int u_\epsilon)^{1+\epsilon}} - \lambda \\ &\leq \frac{\int_{\mathbb{R}^N} |\nabla u_\epsilon|^{1+\epsilon}}{\int u_\epsilon} - \lambda \\ &\leq \frac{m_\epsilon(\lambda)}{\int_\Omega u_\epsilon}. \end{aligned}$$

The limit of the last quantity on the right is negative, finally ∇w_ϵ is bounded in $L^{1+\epsilon}(\Omega)$, in particular w_ϵ is bounded in $BV(\mathbb{R}^N)$, hence one may extract from it a subsequence, still denoted w_ϵ, such that $w_\epsilon \rightharpoonup w$ in $BV(\mathbb{R}^N)$ weakly. Since q is subcritical, one has $\int_\Omega f w^q = 0$ and by lower semicontinuity

$$\int_{\mathbb{R}^N} (|\nabla w| - \lambda w) \leq 0.$$

This contradicts the choice of λ since $\lambda \in [\lambda_1, \lambda_1 + \lambda_q^\star[$, and $|w|_1 = 1$. From this one can conclude that (u_ϵ) is bounded in $L^1(\Omega)$.

As we did for $m_\epsilon(\lambda)$ and $m_q(\lambda)$, one can prove that $p_q(\lambda) = \lim_{\epsilon\to 0} p_\epsilon(\lambda)$. In order to see that $p_q(\lambda) > 0$, let us argue as in the proof of Theorem 1. One has for $\epsilon > 0$ given

$$(\lambda - \lambda_1) \int_\Omega \frac{\phi^q}{(u+\epsilon)^{q-1}} + p_q(\lambda) \int_\Omega f \frac{\phi^q u^{q-1}}{(u+\epsilon)^{q-1}} \leq 0.$$

Using the Lebesgue dominated convergence theorem, one obtains after passing to the limit when ϵ goes to zero that

$$(\lambda - \lambda_1) \int_\Omega \frac{\phi^q}{u^{q-1}} + p_q(\lambda) \int_\Omega f \phi^q \leq 0.$$

From this one gets that $p_q(\lambda) > 0$ for $\lambda > \lambda_1$. For $\lambda = \lambda_1$ the infimum defining $p_q(\lambda_1)$ cannot be zero, because it would contradicts the constraint $\int_\Omega f\phi^q < 0$. A minimizer for $p_q(\lambda_1)$ provides then a nontrivial solution for eq_{λ_1} (up to a mutiplicative constant).

7 Existence's result in the critical case

We suppose in this section that $f \in \mathcal{C}(\overline{\Omega})$. Let us recall that the critical exponent for the Sobolev embedding of $W^{1,1}(\Omega)$ into $L^k(\Omega)$ is $1^\star = \frac{N}{N-1}$.

Before giving the proof of the existence's results in the critical case, let us recall the concentration compactness lemma of P.L. Lions [26] in a form convenient to our setting :

Lemma 1 *Suppose that (u_n) is a sequence in $BV(\mathbb{R}^N)$ which is compactly supported in a fixed compact $\overline{\Omega}$. Suppose that u_n converges weakly in $BV(\mathbb{R}^N)$ to some function $u \in BV(\mathbb{R}^N)$. Then, there exist a subsequence of u_ϵ, denoted in the same manner, two nonnegative bounded measures μ and ν, compactly supported in $\overline{\Omega}$, some numerable set $\{x_i\}$, $i \in \mathbb{N}$, some sequences (μ_i) and (ν_i) of nonnegative reals such that*

$$|\nabla u_n| \rightharpoonup \mu \geq |\nabla u| + \sum_i \mu_i \delta_{x_i},$$

$$|u_n|^{1^\star} \rightharpoonup \nu = |u|^{1^\star} + \sum_i \nu_i \delta_{x_i},$$

in $BV(\mathbb{R}^N)$ weakly, with

$$\nu_i^{\frac{1}{1^\star}} \leq K(N,1)\mu_i.$$

Remark 10 *Let us note that the weak convergence of $|\nabla u_n|$ towards μ implies, since the measures are compactly supported, that*

$$\int_{\overline{\Omega}} |\nabla u_n| \to \int_{\overline{\Omega}} \mu$$

and also

$$\int_{\overline{\Omega}} |u_n|^{1^\star} \to \int_{\overline{\Omega}} \nu.$$

Let us define

$$\lambda^\star = \inf_{\{u \in BV(\Omega),\ \int_\Omega f|u|^{1^\star}=0, |u|_1=1\}} \{\int_\Omega (|\nabla u| - \lambda_1 |u|) + \int_{\partial\Omega} |u|\}$$

Lemma 2

$$0 < \bar{\lambda} = \underline{\lim}_{q\to 1^\star} \lambda_q^\star \leq \lambda^\star$$

$$\overline{\lim}_{q\to 1^\star} m_q(\lambda) \leq m(\lambda)$$

Proof of lemma 2

To prove that $\lambda^\star > 0$, suppose by contradiction that $\lambda^\star = 0$. Let (u_n) be a minimizing sequence for the problem defining $\lambda^\star$. One can assume as usually that $u_n \geq 0$. It is bounded in $L^1(\Omega)$, hence in $BV(\Omega)$. Extending by a now classical way u_n by zero outside of Ω, the extension, still denoted u_n, is bounded in $BV(\mathbb{R}^N)$ and using standard arguments, one can extract from it a subsequence such that $u_n \rightharpoonup u$, $u \geq 0$ in $BV(\mathbb{R}^N)$ weakly. One has $u = 0$ outside of $\overline{\Omega}$, $\int_\Omega |u| = 1$ and by lower semicontinuity

$$\int_{\mathbb{R}^N} |\nabla u| \leq \underline{\lim}_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|$$

Finally using $\lambda^\star = 0$,

$$\int_{\mathbb{R}^N} (|\nabla u| - \lambda_1 |u|) \leq 0 \tag{15}$$

and then this quantity is zero and equality holds in the previous inequalities. In particular

$$\int_{\mathbb{R}^N} |\nabla u| = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|$$

This implies that strong convergence holds in $L^{1^\star}(\Omega)$ and then, $\int_\Omega f|u_n|^{1^\star} \to \int_\Omega f|u|^{1^\star} = 0$. But from (15) u must be an eigenfunction for the eigenvalue λ_1, a contradiction.
The same process enables us to prove that

$$\underline{\lim}_{q\to 1^\star} \lambda_q^\star > 0.$$

Indeed, suppose by contradiction that $\underline{\lim}_{q\to 1^\star} \lambda_q^\star = 0$. Then, there exists $u_q \in W_0^{1,1}(\Omega)$, $\int_\Omega f u_q^q = 0$, $u_q \geq 0$, $|u_q|_1 = 1$ such that

$$\int_\Omega (|\nabla u_q| - \lambda_1 |u_q|) \to 0$$

The extension of u_q by zero outside of $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$, hence, by extracting a subsequence, one gets that there exists $u \in BV(\mathbb{R}^N)$ such that $u \geq 0$ is a weak limit of u_q, u is zero outside of $\overline{\Omega}$, $\int_\Omega u = 1$, and by lower semicontinuity

$$\int_{\mathbb{R}^N} |\nabla u| - \lambda_1 \int_\Omega |u| \leq 0.$$

One obtains as in the previous proof that

$$\int_{\mathbb{R}^N} |\nabla u| = \lim_{q\to 1^\star} \int_{\mathbb{R}^N} |\nabla u_q|,$$

and then $\int_\Omega f u_q^q = 0 = \int_\Omega f u^{1^\star} = 0$, but the previous identities prove that u is an eigenfunction for the eigenvalue λ_1, a contradiction. To prove that $\overline{\lim}_{q\to 1^\star} \lambda_q^\star \leq \lambda^\star$, let $\delta > 0$ and $u \geq 0$ be in $W_0^{1,1}(\Omega)$ such that $\int_\Omega f u^{1^\star} = 0$, $|u|_1 = 1$, and

$$\int_\Omega |\nabla u| - \lambda_1 \leq \lambda^\star + \delta$$

Let q_n be a sequence converging to $1^\star$, such that $\lambda_{q_n}^\star \to \limsup \lambda_q^\star$, and let us consider the sets

$$\mathbb{N} = \{n, \ , \int_\Omega f u^{q_n} = 0\} \cup \{n, \ \int_\Omega f u^{q_n} < 0\} \cup \{n, \ \int_\Omega f u^{q_n} > 0\}$$

Suppose that there exists an infinite sequence $q_n \to 1^\star$ such that $\int_\Omega f u^{q_n} = 0$, then the result is obvious. If not there exists an infinite sequence $q_n \to 1^\star$ such that $\int_\Omega f u^{q_n} > 0$, either there exists an infinite sequence such that $\int_\Omega f u^{q_n} < 0$. Suppose that we are in the first case and define $\alpha(q_n)$ as the real between 0 and 1, such that

$$\alpha(q_n) \int_\Omega f u^{q_n} + (1 - \alpha(q_n)) \int_\Omega f \phi^{q_n} = 0.$$

Finally define

$$v_n = \frac{(\alpha(q_n) u^{q_n} + (1 - \alpha(q_n)) \phi^{q_n})^{\frac{1}{q_n}}}{|(\alpha(q_n) u^{q_n} + (1 - \alpha(q_n)) \phi^{q_n})^{\frac{1}{q_n}}|_1}.$$

One has $v_n \in BV(\Omega)$, $\int_\Omega f v_n^{q_n} = 0$ and v_n converges strongly in BV towards u . As a consequence

$$\lim_{q_n \to 1^\star} \lambda^\star_{q_n} \leq \lim J_{\lambda_1, r}(v_n) \to J_{\lambda_1}(u) = \lambda^\star$$

Suppose that we are in the second case, hence there exists an infinite sequence q_n, $q_n \to 1^\star$, such that $\int_\Omega f u^{q_n} < 0$. Since f is positive somewhere, let v be in $W_0^{1,1}(\Omega)$ such that $\int_\Omega f v^{1^\star} > 0$. Then, for q sufficiently close to $1^\star$, one has $\int_\Omega f v^q > 0$. Let then $\alpha(q_n)$ as the real between 0 and 1 such that

$$\alpha(q_n) \int_\Omega f u^{q_n} + (1 - \alpha(q_n)) \int_\Omega f v^{q_n} = 0,$$

and define

$$v_n = \frac{(\alpha(q_n) u^{q_n} + (1 - \alpha(q_n)) v^{q_n})^{\frac{1}{q_n}}}{|(\alpha(q_n) u^{q_n} + (1 - \alpha(q_n)) v^{q_n})^{\frac{1}{q_n}}|_1}.$$

One has $v_n \in BV(\Omega)$, v_n tends strongly to u in $BV(\Omega)$, $\int_\Omega f v_n^{q_n} = 0$, and we get the same conclusion as in the first case. Finally one has obtained that $\overline{\lim}_{q \to 1^\star} \lambda^\star_q \leq \lambda^\star$.

To end the proof of Lemma 2, let us note that one always has:

$$m(\lambda) \geq \overline{\lim}_{q \to 1^\star} m_q(\lambda)$$

Indeed, let $\delta > 0$ be given, and u be such that $u \in W_0^{1,1}(\Omega)$, $\int_\Omega f u^{1^\star} = -1$, and

$$\int_{\Omega \cup \partial\Omega} (|\nabla u| - \lambda |u|) \leq m(\lambda) + \delta$$

Since $\int_\Omega f|u|^q \to -1$ when q goes to $1^\star$, $v_q = \dfrac{u}{(-\int_\Omega f|u|^q)^{\frac{1}{q}}}$ converges to u in $BV(\Omega)$ strongly and is admissible for the problem defining $m_q(\lambda)$. The same remarks hold true for $p_q(\lambda)$ and $p(\lambda)$.

Theorem 10 *Suppose that Ω^+, Ω^- are $\neq \emptyset$, that $\int_\Omega f\phi^{1^\star} < 0$ for all non-negative first eigenfunction ϕ, and that λ is close enough to λ_1, then $m(\lambda)$ is achieved and $m(\lambda) = \lim_{q\to 1^\star} m_q(\lambda)$. Suppose that*

$$p(\lambda_1)K(N,1)(\sup f)^{\frac{1}{1^\star}} < 1.$$

Then, for λ close to λ_1, $p(\lambda)$ is achieved and lim $p_q(\lambda) = p(\lambda)$.

Proof of theorem 10
We define

$$\bar{m} = \underline{\lim}_{q\to 1^\star} m_q(\lambda)$$

$\bar{m} \leq m$ by Lemma 2, (possibly $\bar{m} = -\infty$). Let u_q be a nonnegative minimizer for $m_q(\lambda)$. u_q satisfies the p.d.e.

$$-\mathrm{div}\sigma(u_q) - \lambda = -m_q(\lambda) f u_q^{q-1} \tag{16}$$

Suppose for a while that the extension of u_q by zero outside of $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. Then $\bar{m} > -\infty$. By extracting a subsequence, one obtains that $u_q \rightharpoonup u$ in $BV(\mathbb{R}^N)$, $u = 0$ outside of $\overline{\Omega}$. The sequence $\sigma(u_q)$ is bounded in $L^\infty(\Omega)$ with a norm less than 1, hence one may extract from it a subsequence, still denoted $\sigma(u_q)$, such that $\sigma(u_q)$ tends weakly in $L^\infty(\Omega)$ towards σ. σ has L^∞ norm less than 1 and one has

$$-\mathrm{div}(\sigma) - \lambda = -\bar{m} f u^{q-1}. \tag{17}$$

Let μ , ν be bounded measures on $\mathbb{R}^N$, $x_i \in \mathbb{R}^{\mathbb{N}}$, μ_i and ν_i as given in Lemma 1. Let us multiply equation (16) by $u_q\varphi$, where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, and integrate over Ω. Passing to the limit, one obtains

$$\int_\Omega (\mu\varphi - \lambda u\varphi) - \int_\Omega \sigma.\nabla\varphi u = -\bar{m}\Big(\int_\Omega f u^{1^\star}\varphi + \sum_i \nu_i f(x_i)\varphi(x_i)\Big). \tag{18}$$

On the other hand, multiplying equation (17) by $u\varphi$, integrating over Ω and substracting the result to (18), one obtains

$$\mu - \sigma.\nabla u = -\bar{m}\sum_i \nu_i f(x_i)\delta_i \tag{19}$$

where the equation holds in $\overline{\Omega}$.

Let us remark here that when $u \in BV$ and $N \geq 2$ the measure ∇u is orthogonal to any Dirac mass (see for example [13]). Let then

$$\mu = g|\nabla u| + \mu^\perp$$

be the Lebesgue Radon Nykodym decomposition of μ into a measure absolutely continuous with respect to $|\nabla u|$ and a measure orthogonal to it. Let us observe that $g \geq 1$, and $\mu^\perp \geq \sum_i \mu_i \delta_i$, more precisally

$$\begin{cases} \mu^\perp = -\bar{m}\sum_i \nu_i f(x_i)\delta_i \\ g|\nabla u| = \sigma.\nabla u \end{cases}$$

One finally gets that

$$\mu^\perp = \sum_i \mu_i' \delta_i = -\bar{m}\sum_i \nu_i f(x_i)\delta_{x_i}$$

with $\mu_i' \geq \mu_i$. On another hand $|\sigma|_\infty \leq 1$ implies that $(g-1)|\nabla u| = 0$ and $\mu = \lim_{q\to 1^\star} |\nabla u_q| = |\nabla u| + \sum_i \mu_i' \delta_i$. One also has

$$\int_\Omega f u^{1^\star} + \sum_i f(x_i)\nu_i = -1$$

and

$$\mu_i' = -\bar{m}\nu_i f(x_i).$$

As a consequence if $f(x_i) < 0$, μ_i', μ_i are zero, and also $\nu_i = 0$, so we are done. Then $f(x_i) \geq 0$ for all i and then
$\int_\Omega f u^{1^\star} = -1 - \sum_i \nu_i f(x_i) \leq -1$,

$$m(\lambda)\left(-\int_\Omega f u^{1^\star}\right)^{\frac{1}{1^\star}} \leq \int_{\Omega\cup\partial\Omega} |\nabla u| - \int_\Omega \lambda u = \bar{m} - \sum_i \mu_i' = \bar{m}\left(-\int_\Omega f u^{1^\star}\right).$$

From this one obtains that

$$(-\bar{m})\left(-\int_\Omega f u^{1^\star}\right) \leq -m(\lambda)\left(-\int_\Omega f u^{1^\star}\right)^{\frac{1}{1^\star}}.$$

This implies that $\bar{m} = m(\lambda)$ and $\int_\Omega f u^{1^\star} = -1$. Finally u realizes the minimum in $m(\lambda)$.

There remains to prove that u_q is bounded in $BV(\mathbb{R}^N)$. For that aim, we prove the claim

Claim: One has $\lim_{\lambda \to \lambda_1} m(\lambda) = 0$.

Proof of the claim :

One already has $m(\lambda) \leq m(\lambda_1)$ for $\lambda > \lambda_1$. Now let u_λ be in $W_0^{1,1}(\Omega)$, $u_\lambda \geq 0$, such that $\int_\Omega f u_\lambda^{1^\star} = -1$ and

$$J_\lambda(u_\lambda) \leq m(\lambda) + (\lambda - \lambda_1)$$

Suppose that $|u_\lambda|_1 \to +\infty$, and define $w_\lambda = \frac{u_\lambda}{|u_\lambda|_1}$. w_λ is bounded, nonnegative and belongs to $W_0^{1,1}(\Omega)$. Extending it by zero outside $\overline{\Omega}$, extracting from it a subsequence, one obtains that there exists $w \in BV(\mathbb{R}^N)$, $w \geq 0$, such that $w = 0$ outside of $\overline{\Omega}$, $|w|_1 = 1$. Using lower semicontinuity, one has

$$0 \leq \int_{\mathbb{R}^N} |\nabla w| - \lambda_1 \leq \underline{\lim}_{\lambda \to \lambda_1} \int_{\mathbb{R}^N} |\nabla w_\lambda| - \lambda \leq 0$$

This implies both that $\int_{\overline{\Omega}} |\nabla w| = \lambda_1$ and $\lim_{\lambda \to \lambda_1} \int_{\overline{\Omega}} |\nabla w_\lambda| = \int_{\overline{\Omega}} |\nabla w|$. Then the convergence of w_λ is strong in $L^{1^\star}(\Omega)$, and $\int_\Omega f w^{1^\star} = 0$. This contradicts the assumption on the eigenfunctions for the eigenvalue λ_1.

Finally u_λ is bounded in $W_0^{1,1}(\Omega)$ and by extracting from it a subsequence, using lower semicontinuity one gets that

$$0 \leq \int_{\mathbb{R}^N} |\nabla u| - \lambda_1 \int_\Omega |u| \leq m(\lambda_1) = 0$$

This yields the desired result.

As a consequence of the claim one may choose λ close enough to λ_1 in order that

$$|m(\lambda)| K(N,1) \sup f^{\frac{1}{1^\star}} < 1.$$

Let us prove under this assumption that (u_q) is bounded in $L^1(\Omega)$. If not, $|u_q|_1$ tends to infinity. Define then $w_q = \frac{u_q}{|u_q|_1}$ The sequence (w_q) is bounded in $BV(\mathbb{R}^N)$, hence one can extract from it a subsequence such that $w_q \rightharpoonup w$ in $BV(\mathbb{R}^N)$. One has $w = 0$ outside of $\overline{\Omega}$, $|w|_1 = 1$, and

$$\int_{\mathbb{R}^N} (|\nabla w| - \lambda |w|) \leq 0.$$

Suppose first that $\int_\Omega fw^{1^\star} = 0$, then, this would contradict the choice of λ with respect to $\lambda^\star$. If one had $\int_\Omega fw^{1^\star} > 0$, one would have

$$p(\lambda)\left(\int_\Omega fw^{1^\star}\right)^{\frac{1}{1^\star}} \leq \int_{\Omega\cup\partial\Omega} |\nabla w| - \lambda \int_\Omega |w| \leq 0,$$

a contradiction, since $p(\lambda) > 0$.

Finally suppose that $\int_\Omega fw^{1^\star} < 0$. Applying the concentration compactness principle of P.L. Lions, [26] there exist some nonnegative measures μ and ν, some numerable set x_i, $x_i \in \overline{\Omega}$, some nonnegative numbers μ_i and ν_i such that

$$|\nabla w_\lambda| \rightharpoonup \mu \geq |\nabla w| + \sum_i \mu_i \delta_{x_i}$$

in $BV(\mathbb{R}^N)$ weakly, and

$$|u_\lambda|^{1^\star} \rightharpoonup \nu = |w|^{1^\star} + \sum_i \nu_i \delta_{x_i}$$

in $BV(\mathbb{R}^N)$ weakly, with

$$\nu_i^{\frac{1}{1^\star}} \leq K(N,1)\mu_i.$$

In particular

$$m(\lambda)(-\int_\Omega fw^{1^\star})^{\frac{1}{1^\star}} \leq \int_{\mathbb{R}^N} |\nabla w| - \lambda \int_\Omega |w| \leq -\sum_i \mu_i.$$

Let us note that one also has

$$\int_\Omega fw^{1^\star} + \sum_i \nu_i f(x_i) = 0.$$

As a consequence one gets

$$m(\lambda)(\sum_i \nu_i f(x_i))^{\frac{1}{1^\star}} \leq -\sum_i \mu_i,$$

hence

$$\sum_i \mu_i \leq -m(\lambda)(\sum_i \nu_i f(x_i))^{\frac{1}{1^\star}} \leq -m(\lambda) \sup f^{\frac{1}{1^\star}} K(N,1) \sum_i \mu_i.$$

By the assumption on $-m(\lambda)$ one obtains that $\sum_i \mu_i = 0$, hence $\nu_i = 0$ and also $\int_\Omega fw^{1^\star} = 0$, once more a contradiction. One has finally obtained that (u_q) is bounded in L^1 and then in $BV(\mathbb{R}^N)$.

We now prove the second part of Theorem 10

Since $p(\lambda)$ decreases when λ increases, supposing that

$$p(\lambda_1)K(N,1)\sup f^{\frac{1}{1^\star}} < 1,$$

the same inequality holds with λ in place of λ_1. Let us define

$$\bar{p} = \overline{\lim}_{q\to 1^\star} p_q(\lambda) \leq p(\lambda)$$

Let us note that $\bar{p}$ is a real nonnegative number. Let u_q be a solution for $p_q(\lambda)$. u_q satisfies the following p.d.e.

$$\left\{ \begin{array}{ccc} -\mathrm{div}\sigma(u_q) - \lambda & = & p_q(\lambda) f u_q^{q-1} \quad u_q \geq 0 \text{ in } \Omega, \\ \sigma(u_q).\nabla u_q & = & |\nabla u_q|, \text{ on } \Omega \cup \partial\Omega \end{array} \right. \tag{20}$$

Following the first part of the proof, one can prove that u_q is bounded in $BV(\mathbb{R}^N)$. Extracting from it a subsequence, and passing to the limit in the p.d.e. satisfied by u_q, one gets that there exists $u \in BV(\mathbb{R}^N)$, u iqs nonnegative, is zero outside of $\overline{\Omega}$ and there exists σ, weak limit of $\sigma(u_q)$ in L^∞, $|\sigma| \leq 1$, such that the equation

$$-\mathrm{div}\sigma - \lambda = \bar{p} f u^{1^\star - 1} \tag{21}$$

holds. By the concentration compactness lemma of P.L.Lions, there exist some nonnegative measures μ and ν, some numerable set x_i, $x_i \in \overline{\Omega}$, some nonnegative numbers μ_i and ν_i such that

$$|\nabla u_q| \rightharpoonup \mu \geq |\nabla u| + \sum_i \mu_i \delta_{x_i}$$

in $BV(\mathbb{R}^N)$ weakly, and

$$|u_q|^{1^\star} \rightharpoonup \nu = |u|^{1^\star} + \sum_i \nu_i \delta_{x_i}$$

in $BV(\mathbb{R}^N)$ weakly, with

$$\nu_i^{\frac{1}{1^\star}} \leq K(N,1)\mu_i.$$

Multiply equation (20) by $u_q\varphi$ with $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and mutiply equation (21) by $u\varphi$. Substracting the two, one gets

$$\mu - \sigma.\nabla u = \bar{p} \sum_i \nu_i f(x_i) \delta_i. \tag{22}$$

Let us remark first that $\bar{p} \neq 0$. If $\bar{p} = 0$ one would have

$$-\text{div}\sigma - \lambda = 0$$

and since $\mu = \sigma.\nabla u$ on $\Omega \cup \partial\Omega$ and $|\sigma|_\infty \leq 1$, one has $\sigma.\nabla u = |\nabla u|$ on $\Omega \cup \partial\Omega$, hence from (22), $|\nabla u_q| \rightharpoonup |\nabla u|$ on $\mathbb{R}^N$ weakly. From this one gets that $\int_\Omega f u^{1^\star} = 1$, and u is an eigenfunction for some λ. If $\lambda > \lambda_1$, since u is nonnegative, this contradicts the results in section 1. If $\lambda = \lambda_1$ the condition $\int_\Omega f u^{1^\star} = 1$ is a contradiction.

Finally $\bar{p} \neq 0$. By the Lebesgue decomposition, one can write

$$\mu = g|\nabla u| + \mu^\perp$$

with $g \geq 1$ and $\mu^\perp$ a measure orthogonal to ∇u. Then, one derives from (22) that

$$\begin{cases} \mu^\perp = \bar{p} \sum_i \nu_i f(x_i) \delta_i, \\ g|\nabla u| = \sigma.\nabla u. \end{cases}$$

One finally gets that

$$\mu^\perp = \sum_i \mu_i' \delta_i$$

for some $\mu_i' \geq \mu_i$. On another hand $|\sigma|_\infty \leq 1$ implies that $(g-1)|\nabla u| = 0$ and $\mu = \lim_{q \to p^\star} |\nabla u_q| = |\nabla u| + \sum_i \mu_i' \delta_i$. Suppose that $f(x_i) < 0$, then, the previous calculation implies that $\mu_i' = 0$, hence $\mu_i = 0$ and ν_i also. Define $\nu_f = \sum_i \nu_i f(x_i)$. By the previous remark $\nu_f \geq 0$. Let us prove that $\nu_f \in [0, 1 + \left(\frac{-m}{\bar{p}}\right)^{\left(\frac{1}{1-\frac{1}{1^\star}}\right)}] = [0, \gamma]$. For that aim, let us recall the relation obtained by passing to the limit in the relation $\int_\Omega f u_q^q = 1$.

$$\int_\Omega f u^{1^\star} + \nu_f = 1.$$

If $\int_\Omega f u^{1^\star} \geq 0$ $\nu_f \in [0, 1]$. If $\int_\Omega f u^{1^\star} < 0$, then

$$m(\lambda)(-\int_\Omega f u^{1^\star})^{\frac{1}{1^\star}} \leq \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| + \int_\Omega (-\lambda)|u| = \bar{p}(\int_\Omega f u^\star)$$

From this one gets

$$\bar{p}(-\int_\Omega fu^{1^\star}) \le -m(\lambda)(-\int fu^{1^\star})^{\frac{1}{1^\star}}.$$

This yields the desired result, using $\nu_f = 1 - \int_\Omega fu^{1^\star}$. Since $\nu_i f(x_i) \ge 0$ for all i, one has $\nu_i f(x_i) \le \gamma$ for all i, and then

$$\begin{aligned}
\mu_i &\le \bar{p}\nu_i f(x_i) \\
&\le \bar{p}\frac{\nu_i f(x_i)}{\gamma}\gamma \\
&\le \bar{p}\left(\frac{\nu_i f(x_i)}{\gamma}\right)^{1-\frac{1}{1^\star}} (\nu_i f(x_i))^{\frac{1}{1^\star}}\gamma^{1-\frac{1}{1^\star}} \\
&\le \bar{p}K(N,1)\sup f^{\frac{1}{1^\star}}\mu_i\gamma^{1-\frac{1}{1^\star}}
\end{aligned}$$

Using $\gamma^{1-\frac{1}{1^\star}} = \left(1 + \left(\frac{-m}{\bar{p}}\right)^{\left(\frac{1}{1-\frac{1}{1^\star}}\right)}\right)^{1-\frac{1}{1^\star}} \le (1 + \frac{-m}{\bar{p}})$, one finally gets

$$\mu_i \le \bar{p}(1 + \frac{-m}{\bar{p}})K(N,1)\sup f^{\frac{1}{1^\star}}\mu_i \le \delta\mu_i$$

for some $\delta < 1$. One obtains that $\mu_i = 0$, hence $\nu_i = 0$, the convergence of u_q towards u in $BV(\mathbb{R}^N)$ is tight and then $\int_\Omega fu^{1^\star} = 1$, $\bar{p} = p(\lambda)$, u is a minimizer for $p(\lambda)$ and it satisfies eq_λ, up to a constant.

8 Appendix : Some complements on weakly almost 1-harmonic functions

We propose here to give a quick proof of Proposition 7 and 8 that we recall here

Proposition 16 *Suppose that Ω is a bounded open set in $\mathbb{R}^N$, which is piecewise $\mathcal{C}^1$ and that u is almost* **1***-harmonic on Ω, then $u \in L^t(\Omega)$ for all $t < \infty$. If in addition $-\mathrm{div}(\sigma) = f \in L^q(\Omega)$ for some $q > N$, then $u \in L^\infty(\Omega)$.*

Proposition 17 *Suppose that u and ϕ are nonnegative in $BV(\Omega) \cap L^\infty(\Omega)$, and are almost weakly 1 harmonic in Ω. Then for all $\epsilon > 0$, and for all*

$k \geq 1$, $q \geq 1$,

$$(\sigma(u) - \sigma(\phi)).\nabla \left(\frac{\phi^k}{(u+\epsilon)^{q-1}} \right) \leq 0$$

Proof of Proposition 7

Let us define for $M > 0$ the truncation u^M as

$$u^M = \begin{cases} u & \text{if } |u| \leq M \\ \frac{Mu}{|u|} & \text{if } |u| \geq M \end{cases}$$

One can observe that u^M is almost $\mathbf{1}$-harmonic and $\sigma(u).\nabla u^M = |\nabla u^M|$ in $\Omega \cup \partial\Omega$. Indeed, let u_ϵ be as in Proposition3. One has

$$\sigma_\epsilon(u_\epsilon).\nabla u_\epsilon^M = |\nabla(u_\epsilon^M)|^{1+\epsilon}$$

in Ω. The sequence u_ϵ^M is bounded in $W^{1,1+\epsilon}(\Omega)$. It converges to u^M in $L^1(\Omega)$, hence $u^M \in BV(\Omega)$ and $u_\epsilon^M \rightharpoonup u^M$ in BV weakly. Using lower semi-continuity,

$$\begin{aligned} |\nabla u^M| &\leq \underline{\lim}_{\epsilon\to 0} |\nabla(u_\epsilon^M)|^{1+\epsilon} \\ &= \underline{\lim}_{\epsilon\to 0} \sigma_\epsilon(u_\epsilon).\nabla u_\epsilon^M \\ &= \sigma(u).\nabla u^M, \end{aligned}$$

the last equality being a consequence of generalized Green's formula in Proposition 1. Since $|\sigma(u)| \leq 1$, one has obtained that $\sigma(u).\nabla u^M = |\nabla u^M|$ in Ω. Suppose now that $x \in \partial\Omega$ is such that $u(x) \neq 0$, then $\sigma(u).\vec{n}(x) = -1$ and also

$$\sigma(u)(x).\vec{n} u^M(x) = -|u^M(x)|$$

Let u be a solution of $-\mathrm{div}(\sigma) = f \in L^q(\Omega)$, where $\sigma(u).\nabla u = |\nabla u|$ in $\Omega \cup \partial\Omega$, multiply the equation by $|u^M|^{1^\star -1} u^M$ and integrate over Ω. One obtains using Proposition 6

$$\int_\Omega |\nabla(|u^M|^{1^\star-1} u^M)| + \int_{\partial\Omega} |u^M|^{1^\star} = \int_\Omega f |u^M|^{1^\star -1} u^M.$$

Let C be some Poincare's constant for the Sobolev embedding from $W_0^{1,1}(\Omega)$ into $L^{1^\star}(\Omega)$. Then, for all $u \in BV(\Omega)$,

$$\left(\int_\Omega |u|^{1^\star} \right)^{\frac{1}{1^\star}} \leq C \left(\int_\Omega |\nabla u| + \int_{\partial\Omega} |u| \right)$$

Let m be some positive number and G_m be defined as

$$G_m = \{x \in \Omega, |f(x)| \geq m\}.$$

Choose m large enough in order that

$$|f|_{L^N(G_m)} \leq \frac{1}{2C}$$

Then one gets

$$\left(\int |u^M|^{(1^\star)^2}\right)^{\frac{1}{1^\star}} \leq C\left(m\int_{G-G_m} |u^M|^{1^\star} + \frac{1}{2C}(\int |u^M|^{(1^\star)^2})^{\frac{1}{1^\star}}\right)$$

One obtains letting M go to infinity that $u \in L^{(1^\star)^2}(\Omega)$, with

$$\frac{1}{2}\left(\int_\Omega |u|^{(1^\star)^2}\right)^{\frac{1}{1^\star}} \leq Cm\int_\Omega |u|^{1^\star}$$

Iterating this process, one gets that $|u|^{(1^\star)^n} \in L^1(\Omega)$ for all n, and then $u \in L^t(\Omega)$ for all $t < \infty$.

We suppose now that $f \in L^q$ for some $q > N$. Let us multiply the equation $-\text{div}(\sigma) = f$ by $|u^M|^{k-1}u^M$, where u^M denotes the truncation of u at the level M. Let k be some integer > 1. One obtains after integrating over Ω

$$\int_\Omega |\nabla(|u^M|^{k-1}u^M)| + \int_{\partial\Omega} |u^M|^k = \int_\Omega f|u^M|^{k-1}u^M$$

Let q' be the conjugate of q. Using Poincare's inequality one gets

$$\left(\int_\Omega ||u^M|^{kq'}|^{\frac{1^\star}{q'}}\right)^{\frac{q'}{1^\star}} \leq \frac{|f|_q}{C}\left(\int_\Omega |u^M|^{kq'}\right)^{\frac{1}{q'}}$$

with some universal constant C. Taking $k = \frac{1^\star}{q'}$ and passing to the limit when $M \to +\infty$ one gets both that $u \in L^{\frac{(1^\star)^2}{q'}}$ and $|u|^{\frac{1^\star}{q'}} \in BV(\Omega)$ with

$$\left(\int_\Omega ||u|^{1^\star}|^{\frac{1^\star}{q'}}\right)^{\frac{1}{1^\star}} \leq \left(\frac{|f|_q}{C}\right)^{q'}\left(\int_\Omega |u|^{1^\star}\right)$$

Iterating this process, and defining

$$\alpha_n = \left(\int_\Omega |u|^{1^\star\left(\frac{1^\star}{q'}\right)^n}\right)^{\left(\frac{q'}{1^\star}\right)^n}$$

one gets

$$\alpha_{n+1} \leq \beta^{K^{n-1}} \alpha_n$$

with $\beta = \left(\frac{|f|_q}{C}\right)^{q'}$ and $K = \frac{q'}{1^\star} < 1$. From this one obtains

$$\alpha_{n+1} \leq \beta^{\frac{K^n - 1}{K-1}} \alpha_0.$$

Finally $u \in L^\infty$ and

$$|u|_\infty^{1^\star} \leq \beta^{\frac{1}{1-K}} |u|_{1^\star}^{1^\star}.$$

Proof of Proposition 8 .

Let ϕ_ϵ and u_ϵ, as in the proof of Proposition 3. Denote by ϕ_ϵ^+ and u_ϵ^+ the positive parts of ϕ_ϵ and u_ϵ. Using Young's inequality one has

$$A_\epsilon = k\frac{(\phi_\epsilon^+)^{k-1}}{(u_\epsilon^+ + \lambda)^{q-1}} \left(\frac{1}{1+\epsilon}|\nabla \phi_\epsilon|^{1+\epsilon} - \nabla \phi_\epsilon . \sigma_\epsilon(u_\epsilon) + \frac{\epsilon}{1+\epsilon}|\nabla u_\epsilon|^{1+\epsilon}\right) \geq 0$$

and

$$B_\epsilon = (q-1)\frac{(\phi_\epsilon^+)^{k}}{(u_\epsilon^+ + \lambda)^{q}} \left(\frac{1}{1+\epsilon}|\nabla u_\epsilon|^{1+\epsilon} - \nabla u_\epsilon . \sigma_\epsilon(\phi_\epsilon) + \frac{\epsilon}{1+\epsilon}|\nabla \phi_\epsilon|^{1+\epsilon}\right) \geq 0$$

(If $k = 1$, replace ϕ_ϵ^+ by ϕ_ϵ.) On the other hand one has

$$\begin{aligned}
(\sigma_\epsilon(\phi_\epsilon) - \sigma_\epsilon(u_\epsilon)) \quad &. \quad \nabla\left(\frac{(\phi_\epsilon^+)^k}{(u_\epsilon^+ + \lambda)^{q-1}}\right) = \frac{k(\phi_\epsilon^+)^{k-1}}{(u_\epsilon^+ + \lambda)^{q-1}}\left(|\nabla \phi_\epsilon|^{1+\epsilon} - \sigma_\epsilon(u_\epsilon).\nabla \phi_\epsilon\right) \\
&- \quad (1-q)\frac{(\phi_\epsilon^+)^k sign^+ u_\epsilon}{(u_\epsilon^+ + \lambda)^q}\left(|\nabla u_\epsilon|^{1+\epsilon} - \sigma_\epsilon(\phi_\epsilon).\nabla u_\epsilon\right) \\
&= \quad A_\epsilon + B_\epsilon sign^+ u_\epsilon \\
&- \quad \frac{\epsilon}{1+\epsilon}\left(|\nabla \phi_\epsilon|^{1+\epsilon} + |\nabla u_\epsilon|^{1+\epsilon}\right)\left(\frac{k(\phi_\epsilon^+)^{k-1}}{(u_\epsilon^+ + \lambda)^{q-1}} + \frac{(q-1)(\phi_\epsilon^+)^k sign^+ u_\epsilon}{(u_\epsilon^+ + \lambda)^q}\right)
\end{aligned}$$

Using the fact that $\dfrac{k(\phi_\epsilon^+)^{k-1}}{(u_\epsilon^+ + \lambda)^{q-1}} + \dfrac{(q-1)(sign^+ u_\epsilon)(\phi_\epsilon^+)^k}{(u_\epsilon^+ + \lambda)^q}$ is bounded in $L^\infty(\Omega)$ and $|\nabla \phi_\epsilon|^{1+\epsilon} + |\nabla u_\epsilon|^{1+\epsilon}$ is bounded in $L^1(\Omega)$ one gets that the weak limit in $BV(\Omega)$ of $(\sigma_\epsilon(\phi_\epsilon) - \sigma_\epsilon(u_\epsilon)).\nabla(\dfrac{(\phi_\epsilon^+)^k}{(u_\epsilon^+ + \lambda)^{q-1}})$ is nonnegative. Observe now that the sequence $\dfrac{(\phi_\epsilon^+)^k}{(u_\epsilon^+ + \lambda)^{q-1}}$ tends to $\dfrac{\phi^k}{(u+\lambda)^{q-1}}$ in $(BV \cap L^\infty)(\Omega)$ weakly. Since $\sigma_\epsilon(\phi_\epsilon) - \sigma_\epsilon(u_\epsilon)$ tends to $\sigma(\phi) - \sigma(u)$ in L^q for all $q < \infty$

and $\mathrm{div}\sigma_\epsilon(\phi_\epsilon) - \mathrm{div}\sigma_\epsilon(u_\epsilon)$ tends to $\mathrm{div}(\sigma(\phi) - \sigma(u))$ in L^N, one gets that $\left(\sigma_\epsilon(\phi_\epsilon) - \sigma_\epsilon(u_\epsilon)\right).\nabla\left(\frac{(\phi_\epsilon^+)^k}{(u_\epsilon^+ + \lambda)^{q-1}}\right)$ tends to $(\sigma(\phi) - \sigma(u)).\nabla\left(\frac{\phi^k}{(u+\lambda)^{q-1}}\right)$. One has obtained the result.

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